MATH 211 | Test #1 | October 4, 2016

Name: Richard | R. Hammack | Score:______

Directions Unless noted otherwise, you must show and explain your work to get full credit.

1. (4 points) **Short answer.** $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

   (a) Express the set $\{3x + 1 : x \in \mathbb{N}\}$ by listing its elements between braces.

   $\boxed{\{4, 7, 10, 13, 16, 19, 22, 25, \ldots\}}$

   (b) Write $\{\ldots \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8 \ldots\}$ in set-builder notation.

   $\boxed{\{2^n : n \in \mathbb{Z}\}}$

2. (10 points) **Short answer.** In this problem $A = \{1, 2, 3\}$ and $B = \{1, 4\}$.

   (a) $A \cap B = \boxed{\{1\}}$

   (b) $(A - B) \times B = \{2, 3\} \times \{1, 4\} = \boxed{\{(2,1), (2,4), (3,1), (3,4)\}}$

   (c) $\mathscr{P}(B) = \boxed{\{\emptyset, \{1\}, \{4\}, \{1,4\}\}}$

   (d) $\mathscr{P}(B) - \mathscr{P}(A) = \boxed{\{\{4\}, \{1,4\}\}}$

   (e) How many subsets of cardinality 4 does $A \times B$ have? $|A \times B| = 3 \cdot 2 = 6$, so the number of subsets of $A \times B$ that have cardinality 4 is $\binom{6}{4} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5 \cdot 4!}{4!\,2!} = \frac{6 \cdot 5}{2} = \boxed{15}$

3. (6 points) **Short answer.** Suppose $A = \{1, 3, 4, 6, 9\}$ and $B = \{4, 5, 6, 8, 9\}$ are two sets in a universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   (a) $\overline{A \cap B} = \overline{\{4, 6, 9\}} = \boxed{\{1, 2, 3, 5, 7, 8\}}$

   (b) $\overline{A} \cup \overline{B} = \{2, 5, 7, 8\} \cup \{1, 2, 3, 7\} = \boxed{\{1, 2, 3, 5, 7, 8\}}$

4. (20 points) Five cards are dealt off of a standard 52-card deck and lined up in a row.

(a) How many different lineups are possible?

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \boxed{311{,}875{,}200}$$

(multiplication principle)

(b) How many of these lineups have at least one red card?

B B B B B

$$\underbrace{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}_{\text{all lineups}} - \underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{all black}} = \underbrace{\boxed{303{,}981{,}600}}_{\text{at least one red}}$$

(subtraction principle)

(c) How many lineups have exactly one red card?

R B B B B   B R B B B   B B R B B   B B B R B   B B B B R

$$26 \cdot 26 \cdot 25 \cdot 24 \cdot 23 + 26 \cdot 26 \cdot 25 \cdot 24 \cdot 23 + 26 \cdot 25 \cdot 26 \cdot 24 \cdot 23 + 26 \cdot 25 \cdot 24 \cdot 26 \cdot 23 + 26 \cdot 25 \cdot 24 \cdot 23 \cdot 26$$

$$= 5 \cdot 26 \cdot 26 \cdot 25 \cdot 24 \cdot 23 = \boxed{46{,}644{,}000}$$

(addition principle)

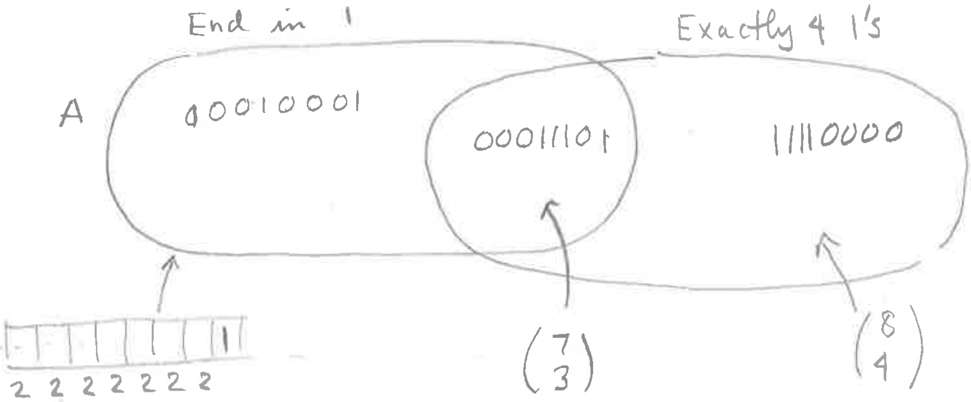
(d) How many lineups are either all black or all hearts?

B B B B B   ♡ ♡ ♡ ♡ ♡

$$\underbrace{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}_{\text{all black}} + \underbrace{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}_{\text{all hearts}} = \boxed{8{,}048{,}040}$$

(addition principle)

5. (10 points) How many 8-digit binary strings end in 1 or have exactly four 1's?



(choose 3 of first 7 positions for 1's) (choose 4 of 8 positions for 1's)

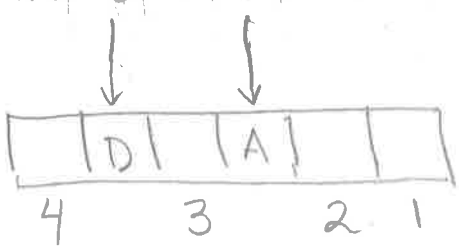
Note: $A \cap B$ consists of all 8-digit strings that end in 1 and have exactly four 1's

By inclusion-exclusion principle, the answer is

$$|A \cup B| = |A| + |B| - |A \cup B| = 2^7 + \binom{8}{4} - \binom{7}{3} = \boxed{163}$$

6. (10 points) This problem concerns length-6 lists made from the symbols A, B, C, D, E, F, without repetition. How many such lists are there in which the D occurs before the A?

First choose 2 of 6 positions for D and A: $\binom{6}{2}$ ways



Then fill in remaining slots with B, C, E, F: $4!$ ways.

<u>Answer</u> $$\binom{6}{2}4! = \frac{6!}{2!\,4!}4! = \frac{6!}{2!} = 6\cdot5\cdot4\cdot3$$

$$= \boxed{360}$$

7. (10 points) How many permutations of the letters A, B, C, D, E, F, G are there in which the letters ABC appear consecutively, in alphabetical order?

$$\underbrace{ABC\ \boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }}_{4\cdot 3\cdot 2\cdot 1} \leftarrow 4!$$

$$\underbrace{\boxed{\ }}_{4}\ ABC\ \underbrace{\boxed{\ }\boxed{\ }\boxed{\ }}_{3\ 2\ 1} \leftarrow 4!$$

$$\underbrace{\boxed{\ }\boxed{\ }}_{4\ 3}\ ABC\ \underbrace{\boxed{\ }\boxed{\ }}_{2\ 1} \leftarrow 4!$$

$$\underbrace{\boxed{\ }\boxed{\ }\boxed{\ }}_{4\ 3\ 2}\ ABC\ \underbrace{\boxed{\ }}_{1} \leftarrow 4!$$

$$\underbrace{\boxed{\ }\boxed{\ }\boxed{\ }\boxed{\ }}_{4\ 3\ 2\ 1}\ ABC \leftarrow 4!$$

These lists fall into five types as shown on the left. Each type has 4! lists by multiplication principle. Now use addition principle to get answer:

Answer $5\cdot 4! = 5! = \boxed{120}$

Note: Also you can treat $\boxed{ABC}$ as a single symbol and get $5! = 120$ permutations of the 5 symbols $\boxed{ABC}$, D, E, F, G.

8. (10 points) In how many ways can you place 20 identical balls into 5 different boxes?

Model this with stars and bars, so stars represent balls in each box and bars divide the piles of balls:

$$\underbrace{*\,*\,*\cdots *}_{\text{Balls in Box 1}}\ \Big|\ \underbrace{*\,*\,*\,*\cdots *}_{\text{Balls in Box 2}}\ \Big|\ \underbrace{*\,*\,*\,*\,*}_{\text{Box 3}}\ \Big|\ \underbrace{*\,*\,*\,*\,*}_{\text{Box 4}}\ \Big|\ \underbrace{*\,*\,*\,*\,*}_{\text{Box 5}}$$

20 stars and 4 bars (list of length 24)

Choose 4 out of 24 positions for bars and fill remaining positions with stars. There are $\binom{24}{4} = \frac{24!}{4!\,20!} = \frac{24\cdot 23\cdot 22\cdot 21\cdot \cancel{20!}}{4\cdot 3\cdot 2\cdot \cancel{20!}} = \boxed{10,626}$

ways to do this.

9. (10 points)

(a) Write out Pascal's triangle to Row 5.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

(b) Use part (a) as an aid in expanding $(2x - y)^5$. $= (2x + (-y))^5 =$

$$1(2x)^5 + 5(2x)^4(-y) + 10(2x)^3(-y)^2 + 10(2x)^2(-y)^3 + 5(2x)(-y)^4 + 1(-y)^5$$

$$= \boxed{32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5}$$

10. (10 points)

(a) State the binomial theorem.

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

(b) Use the binomial theorem to show that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$

$$2^n = (1+1)^n = \binom{n}{0}1^n 1^0 + \binom{n}{1}1^{n-1}1^1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^0 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$