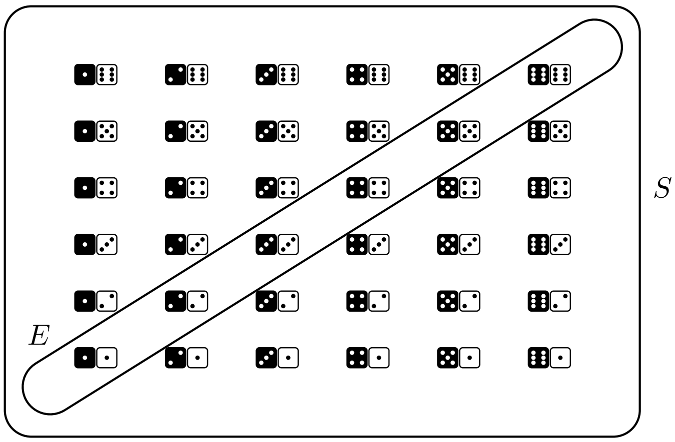
MATH 211 Test #2 ♡ October 31, 2019

Name:______________ R. Hammack Score:______

Directions You must show your work to get full credit. This test is closed-book and closed-notes. No calculators or other electronic devices are allowed. Simplify your answers if it is easy to do so, but you may leave complex answers unsimplified. All you will need is something to write with.

1. (10 points) You have two fair 6-sided dice, a black one and a white one. You toss them both. Write out the sample space S, and indicate the event $E \subseteq S$ of both dice showing the same number (rolling doubles). Find $p(E)$.



$$p(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6} = \boxed{16.\overline{6}\%}$$

2. (10 points) Toss a fair 6-sided dice 10 times in a row.
What are the chances of getting exactly two fives (⚄) among the 10 rolls?

The sample space S is the set of all length-10 lists made from the symbols ⚀, ⚁, ⚂, ⚃, ⚄, ⚅, with repetition allowed. Thus $|S| = 6^{10}$.

Consider the event $E \subseteq S$ of exactly two ⚄'s. To make a list in E, start with 10 empty spots.

Then pick two out of the ten spots for ⚄'s. You can do this in $\binom{10}{2}$ ways.

			⚄			⚄			

Now there are 5 choices for each of the remaining five spots (anything but ⚄).

5	5	5	⚄	5	5	⚄	5	5	5

By the multiplication principle, $|E| = 5^8 \binom{10}{2}$.

Therefore $p(E) = \dfrac{|E|}{|S|} = \dfrac{5^8\binom{10}{2}}{6^{10}} = \dfrac{5^8 \cdot 45}{6^{10}} \approx \boxed{\mathbf{29.071\%.}}$

3. (10 points) A 7-card hand is dealt off a shuffled standard 52-card deck.
What is the probability that the hand consists entirely of red cards, or has no hearts?

The sample space S is the set of all 7-element subsets of the deck of 52 cards, so $|S| = \binom{52}{7}$.

Consider the following events:
A: The hand consists entirely of red cards.
B: The hand has no hearts.

Now, $|A| = \binom{26}{7}$. (The number of ways to choose 7 red cards.)
Now, $|B| = \binom{39}{7}$. (The number of ways to choose 7 non-heart cards.)

Notice that $A \cap B$ are the hands that consist entirely of red cards, but have no hearts.
These are the hands consisting entirely of diamonds. Thus $|A \cap B| = \binom{13}{7}$.

We seek $p(A \cup B) = p(A) + p(B) - P(A \cap B) = \dfrac{|A|}{|S|} + \dfrac{|B|}{|S|} - \dfrac{|A \cap B|}{|S|} = \boxed{\dfrac{\binom{26}{7}}{\binom{52}{7}} + \dfrac{\binom{39}{7}}{\binom{52}{7}} - \dfrac{\binom{13}{7}}{\binom{52}{7}}}.$

4. (10 points) A 7-card hand is dealt off a shuffled standard 52-card deck.
What is the probability that not all of the cards are black?

The sample space S is the set of all 7-element subsets of the deck of 52 cards, so $|S| = \binom{52}{7}$.

Let E be the event of all seven cards being black, so $|E| = \binom{26}{7}$.
(The number of ways to choose 7 out of 26 black cards.)

Now, $\overline{E}$ is the event of not all the seven cards being black. Thus our answer is

$$p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{\binom{26}{7}}{\binom{52}{7}} = 1 - \frac{\frac{P(26,7)}{7!}}{\frac{P(26,7)}{7!}} = 1 - \frac{P(26,7)}{P(52,7)} = 1 - \frac{55}{11186} \approx 99.5\%$$

5. (10 points) A box contains 8 red balls, 4 green balls and 1 blue ball. You reach in and remove two balls, one after the other. What is the probability that the two balls have the same color?

The sample space is $S = \{\, RR,\ RG,\ GR,\ GG,\ RB,\ BR,\ GB,\ BG \,\}$.
Thus the event of both balls having the same color is $E = \{RR, GG\}$.

Let A be the event of the first draw being R.
Let B be the event of the second draw being R.
Let C be the event of the first draw being G.
Let D be the event of the second draw being G.

$P(RR) = p(A \cap B) = p(A) \cdot p(B|A) = \frac{8}{13} \cdot \frac{7}{12}$.
$P(GG) = p(C \cap D) = p(C) \cdot p(D|C) = \frac{4}{13} \cdot \frac{3}{12}$.

Then $p(E) = p\big(\{RR, GG\}\big) = p(RR) + p(GG) = \dfrac{8}{13} \cdot \dfrac{7}{12} + \dfrac{4}{13} \cdot \dfrac{3}{12} = \dfrac{8 \cdot 7 + 4 \cdot 3}{13 \cdot 12} = \dfrac{68}{156} \approx \boxed{\mathbf{43.589\%}}$

6. (10 points) Suppose $A, B \subseteq S$ are two events in the sample space S of some experiment.
Suppose $p(A) = 60\%$, $p(B) = 80\%$ and $p(A|B) = 50\%$.

(a) $p(A \cap B) = p(A|B) \cdot p(B) = (0.5) \cdot (0.8) = 0.4 = \boxed{\mathbf{40\%.}}$

(b) $p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.6 + 0.8 - 0.4 = 1.0 = \boxed{\mathbf{100\%.}}$

(c) $p(B|A) = \dfrac{p(A \cap B)}{p(A)} = \dfrac{0.4}{0.6} = \dfrac{0.2}{0.3} = \dfrac{2}{3} = \boxed{66.\overline{6}\%.}$

(d) $p\left(\overline{B}\right) = 1 - p(B) = 1 - 0.8 = 0.2 = \boxed{\mathbf{20\%.}}$

(e) Are A and B independent or dependent?
As $p(A) = 60\%$ and $p(A|B) = 50\%$, we know $p(A) \neq p(A|B)$. Hence $\boxed{A \text{ and } B \text{ are dependent.}}$

7. (10 points) A man has three children, and there are more girls than boys.
What is the probability that his oldest child is a boy?

The event of having three children, more girls than boys is $E = \{GGG, GGB, GBG, BGG\}$.
Only one of the four outcomes in E has the oldest child as a boy,
So the probability that his oldest child is a boy is $\frac{1}{4} = \boxed{\mathbf{25\%.}}$

8. (10 points) Give the output for the following chunk of pseudocode.

$y := 3$
for $n := 1$ **to** 4 **do**
  output y
  $y := 10 \cdot y$
end

Output: 3 30 300 3000

9. (10 points) What does the following algorithm do?

Algorithm

Input: A natural number $n \in \mathbb{N}$
Output: ?
begin
  while $(\, n > 1 \,)$ **do**
    $n := n - 2$
  end
  if $(\, n = 0 \,)$ **then**
    output "Yes"
  else
    output "No"
  end
end

The input is a natural number n.

The while loop keeps subtracting 2 from n until the result is no longer greater than 1.
Thus after the while loop is finished n is either 0 or 1.

For example for input $n = 8$, the while loop keeps subtracting 2 from n until it gets $n = 0$.
And for input $n = 9$, the while loop keeps subtracting 2 from n until it gets $n = 1$.

In general, if n is even, the the while loop terminates with $n = 0$.
And if n is odd, the the while loop terminates with $n = 1$.

The **if** statement following the loop outputs either Yes or No, depending on whether n is even or odd.

Answer:

The algorithm determines whether n is even or odd. It returns Yes if n is even, and No if n is odd.

10. (10 points) Write an algorithm whose input is a positive integer n and whose output is the first n terms of the sequence $6, 11, 16, 21, 26, 31, 36, 41 \ldots$.

This sequence starts with 6, and each new term is 5 plus the previous term.

Algorithm

Input: A positive integer n.
Output: First n terms of sequence 6, 11, 17, 23, 29, 35, 41...
begin
  $a := 6$
  for $i = 1$ **to** n **do**
    output a .... (this is the ith term)
    $a := a + 5$ .... (now add 5 to the ith term to get the next term)
  end
end
