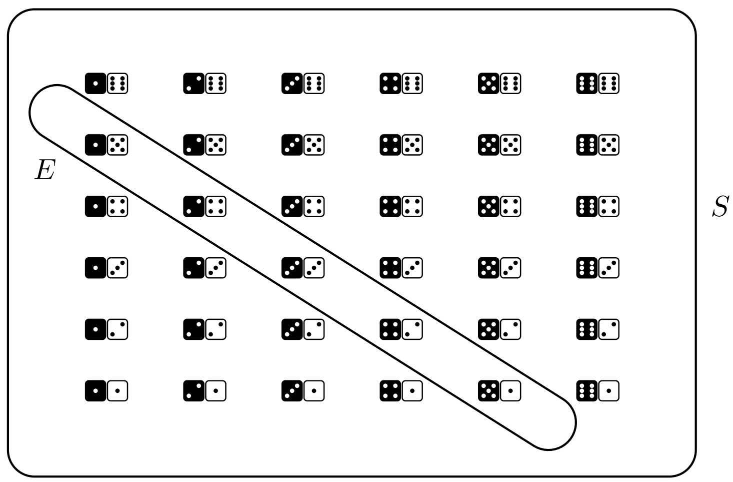
MATH 211 Test #2 $\diamondsuit$ October 31, 2019

Name:______________________ R. Hammack Score:_______

Directions You must show your work to get full credit. This test is closed-book and closed-notes. No calculators or other electronic devices are allowed. Simplify your answers if it is easy to do so, but you may leave complex answers unsimplified. All you will need is something to write with.

1. (10 points) You have two fair 6-sided dice, a black one and a white one. You toss them both. Write out the sample space S, and indicate indicate the event $E \subseteq S$ of the two dice adding to 6. Find $p(E)$.



$$p(E) = \frac{|E|}{|S|} = \frac{5}{36} = \boxed{13.\overline{8}\%}$$

2. (10 points) Toss a fair 6-sided dice 10 times in a row.
What are the chances that at least one of the tosses is even?

The sample space S is the set of all length-10 lists made from the symbols ⚀, ⚁, ⚂, ⚃, ⚄, ⚅, with repetition allowed. Thus $|S| = 6^{10}$.

Let E be the event of *none* of the six tosses being even.
So E is the set of all length-10 lists made from the symbols ⚀, ⚂, ⚄, with repetition allowed.
Thus $|E| = 3^{10}$.

Note that $\overline{E}$ is the event of at least one of the tosses being even.
Thus the answer to our question is $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{3^{10}}{6^{10}} =$

$$1 - \frac{3^{10}}{(2\cdot 3)^{10}} = 1 - \frac{3^{10}}{2^{10}3^{10}} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1024-1}{1024} = \frac{1023}{1024} \approx \boxed{\mathbf{99.902\%}}.$$

3. (10 points) A 7-card hand is dealt off a shuffled standard 52-card deck.
What is the probability that not all of the cards are hearts?

The sample space S is the set of all 7-element subsets of the deck of 52 cards, so $|S| = \binom{52}{7}$.

Let E be the event of all seven cards being hearts, so $|E| = \binom{13}{7}$.
(The number of ways to choose 7 out of 13 heart cards.)

Now, $\overline{E}$ is the event of not all the seven cards being hearts. Thus our answer is

$$p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{\binom{13}{7}}{\binom{52}{7}} = 1 - \frac{\frac{P(13,7)}{7!}}{\frac{P(52,7)}{7!}} = 1 - \frac{P(13,7)}{P(52,7)} = 1 - \frac{132}{10291120} \approx \boxed{\mathbf{99.999998822\%.}}$$

4. (10 points) A 7-card hand is dealt off a shuffled standard 52-card deck.
What is the probability that the hand consists entirely of club cards, or has no hearts?

The sample space S is the set of all 7-element subsets of the deck of 52 cards, so $|S| = \binom{52}{7}$.

Consider the following events.
A: *Hand consists entirely clubs*
B: *Hand has no hearts*

Notice that $A \subseteq B$.

To answer the question, we need to compute $p(A \cup B)$, but since $A \subseteq B$, it follows that $A \cup B = B$.
Therefore to get our answer, we only need to compute $p(B)$.

Now, to get a 7-card hand with no hearts, we just select 7 cards from the 39 non-heart cards, and there are $\binom{39}{7}$ ways to do this. Therefore $|B| = \binom{39}{7}$.

Answer: $p(B) = \frac{\binom{39}{7}}{\binom{52}{7}} = \frac{\frac{P(39,7)}{7!}}{\frac{P(52,7)}{7!}} = \frac{P(39,7)}{P(52,7)} = \frac{39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 33}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46} \approx 0.11496 = \boxed{\mathbf{11.496\%.}}$

5. (10 points) A box contains 7 red balls, 5 green balls and 1 blue ball. You reach in and remove two balls, one after the other. What is the probability that the two balls have the same color? The

sample space is $S = \{\, RR,\ RG,\ GR,\ GG,\ RB,\ BR,\ GB,\ BG \,\}$.
Thus the event of both balls having the same color is $E = \{RR, GG\}$.

Let A be the event of the first draw being R.
Let B be the event of the second draw being R.

Let C be the event of the first draw being G.
Let D be the event of the second draw being G.

$P(RR) = p(A \cap B) = p(A) \cdot p(B|A) = \frac{7}{13} \cdot \frac{6}{12}$.
$P(GG) = p(C \cap D) = p(C) \cdot p(D|C) = \frac{5}{13} \cdot \frac{4}{12}$.

Then $p(E) = p\big(\{RR, GG\}\big) = p(RR) + p(GG) = \dfrac{7}{13} \cdot \dfrac{6}{12} + \dfrac{5}{13} \cdot \dfrac{4}{12} = \dfrac{7 \cdot 6 + 5 \cdot 4}{13 \cdot 12} = \dfrac{62}{156} \approx \boxed{\mathbf{39.74\%}}$

6. (10 points) Suppose $A, B \subseteq S$ are two events in the sample space S of some experiment.
Suppose $p(A) = 50\%$, $p(B) = 60\%$ and $p(A|B) = 50\%$.

(a) $p(A \cap B) = p(A|B) \cdot P(B) = (0.5) \cdot (0.6) = 0.3 = \boxed{\mathbf{30\%}}$

(b) $p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.5 + 0.6 - 0.3 = 0.8 = \boxed{\mathbf{80\%}}$

(c) $p(B|A) = \dfrac{p(A \cap B)}{p(A)} = \dfrac{0.3}{0.5} = \boxed{\mathbf{60\%}}$

(d) $p\left(\overline{B}\right) = 1 - p(B) = 1 - 0.6 = 0.4 = \boxed{\mathbf{40\%}}$

7. (10 points) A woman has four children (no twins). Consider the following events:
A: She has two girls and two boys.
B: Her oldest child is a boy.
Are events A and B independent, dependent, or is there not enough information to say for sure?

Note that $p(B) = 50\%$

Now suppose that A has occurred. Here is the event A:

$$A = \{GGBB, BGGB, BBGG, GBBG, GBGB, BGBG\}.$$

Notice that in three out of six of the outcomes in A, the oldest child is a boy.

Therefore $p(B|A) = \dfrac{|A \cap B|}{|A|} = \dfrac{3}{6} = 50\%.$

Because $p(B) = p(B|A)$, it follows that $\boxed{\textbf{events } A \textbf{ and } B \textbf{ are independent.}}$

8. (10 points) Give the output for the following chunk of pseudocode.

$y := 5$
output y
for $n := 1$ **to** 3 **do**
  $y := 10 \cdot y$
  output y
end

$\boxed{\textbf{Output: } 5 \quad 50 \quad 500 \quad 5000}$

9. (10 points) What does the following algorithm do?

Algorithm

Input: A natural number $n \in \mathbb{N}$
Output: ?
begin
  while $(n > 1)$ **do**
    $n := n - 2$
  end
  if $(n = 0)$ **then**
    output "Yes"
  else
    output "No"
  end
end

The input is a natural number n.

The while loop keeps subtracting 2 from n until the result is no longer greater than 1.
Thus after the while loop is finished n is either 0 or 1.

For example for input $n = 8$, the while loop keeps subtracting 2 from n until it gets $n = 0$.
And for input $n = 9$, the while loop keeps subtracting 2 from n until it gets $n = 1$.

In general, if n is even, the the while loop terminates with $n = 0$.
And if n is odd, the the while loop terminates with $n = 1$.

The **if** statement following the loop outputs either Yes or No, depending on whether n is even or odd.

Answer:

The algorithm determines whether n is even or odd. It returns Yes if n is even, and No if n is odd.

10. (10 points) Write an algorithm whose input is a positive integer n and whose output is the first n terms of the sequence $5, 11, 17, 23, 29, 35, 41 \ldots$.

This sequence starts with 5, and each new term is 6 plus the previous term.

Algorithm

Input: A positive integer n.
Output: First n terms of sequence 5, 11, 17, 23, 29, 35, 41...
begin
  $a := 5$
  for $i = 1$ **to** n **do**
    output a .... (this is the ith term)
    $a := a + 6$ .... (now add 6 to the ith term to get the next term)
  end
end
