MATH 211 FINAL EXAM December 13, 2016

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Directions: Answer each question in the space provided. To get full credit you must show all of your work, unless instructed otherwise. Use of calculators is **not** allowed on this test.

1. (10 points) Write each set by listing its elements between braces.

(a) $\{m \in \mathbb{N} : 3|m\} =$ $\boxed{\{3, 6, 9, 12, 15, \ldots\}}$

(b) $\{x \in \mathbb{R} : x^2 - 2x = 0\} =$ $\boxed{\{0, 2\}}$ ← $x^2 - 2x = 0$, $x(x-2) = 0$, $x = 0 \quad x = 2$

(c) $\mathscr{P}(\{1,2\}) =$ $\boxed{\{\emptyset, \{1\}, \{2\}, \{1,2\}\}}$

(d) $\{1,2\} \times \mathscr{P}(\{1,2\}) =$ $\{(1,\emptyset), (2,\emptyset), (1,\{1\}), (2,\{1\}), (1,\{2\}), (2,\{2\}), (1,\{1,2\}), (2,\{1,2\})\}$

(e) $\{1,2\} \cap \mathscr{P}(\{1,2\}) =$ $\boxed{\emptyset}$ ← Note $1 \notin \mathscr{P}(\{1,2\})$ and $2 \notin \mathscr{P}(\{1,2\})$

2. (6 points)

(a) Suppose the following statement is **false**: $(P \wedge \sim Q) \Rightarrow (R \Rightarrow S)$
Is there enough information given to determine the truth values of P, Q, R and S? If so, what are they?

The only way that $(P\wedge \sim Q) \Rightarrow (R \Rightarrow S)$ can be false is if $(P \wedge \sim Q)$ is true and $(R \Rightarrow S)$ is false. This means

$\boxed{P = T, \ Q = F, \ R = T, \ S = F}$

(b) Write a sentence that is the negation of the following sentence:
There exists a real number a for which $a + x = x$ for every real number x. ← $\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, a + x = x$

Negation: $\sim(\exists a \in \mathbb{R}, \forall x \in \mathbb{R}, a + x = x) = \forall a \in \mathbb{R}, \exists x \in \mathbb{R}, a + x \neq x$.

For any real number a, there is a real number x for which $a + x \neq x$.

(c) Decide if the following statement true or false. Briefly justify answer. $\forall n \in \mathbb{N}, \exists X \in \mathscr{P}(\mathbb{N}), |X| = n - 1$

This is TRUE. If $n \in \mathbb{N}$, then $n \in \{1,2,3,4,\ldots\}$ so $n-1 \in \{0,1,2,3,\ldots\}$. You can certainly find an $X \subseteq \mathbb{N}$ with $|X| = n-1$.

3. (6 points) Write a truth table for $(P \Rightarrow Q) \Leftrightarrow (P \vee Q)$.

P	Q	$P \Rightarrow Q$	$P \vee Q$	$(P \Rightarrow Q) \Leftrightarrow (P \vee Q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

4. (6 points) A 5-card poker hand is called a *flush* if all cards are the same suit. How many different flushes are there?

$\binom{13}{5}$ hands are all ♡'s

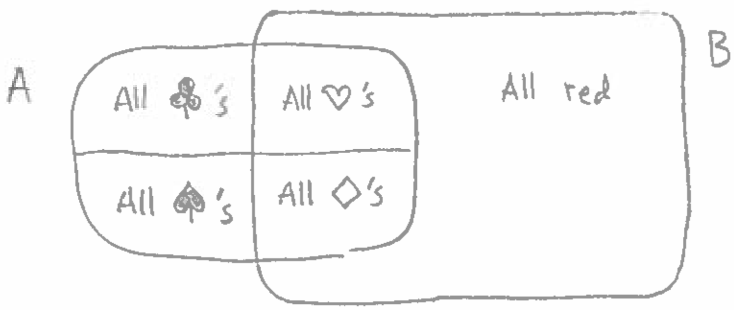
$\binom{13}{5}$ hands are all ◇'s

$\binom{13}{5}$ hands are all ♣'s

$\binom{13}{5}$ hands are all ♠'s

Answer The number of hands that are all the same suit is $4\binom{13}{5} = 4\frac{13\cdot12\cdot11\cdot10\cdot9}{5\cdot4\cdot3\cdot2} = 4\cdot13\cdot11\cdot9 = \boxed{5148}$

5. (6 points) Consider 4-card hands dealt off of a standard 52-card deck. How many hands are there for which all 4 cards are of the same suit or all 4 cards are red?



Let A be the set of 4-card hands where all 4 cards have same suit

Then A = { [♡5 ♡10 ♡7 ♡K], [♣3 ♣7 ♣J ♣2], [◇8 ◇7 ◇K ◇Q], ... }

Let B be the set of 4-card hands where all 4 cards are red

Then B = { [♡5 ♡3 ♡7 ♡K], [♡5 ◇3 ◇7 ♡K], [◇8 ◇7 ◇K ◇Q], ... }

So $A \cap B$ = (set of 4-card hands where either all cards are hearts or all cards are diamonds)

By inclusion-exclusion principle, the answer is

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 4\binom{13}{4} + \binom{26}{4} - 2\binom{13}{4} = 2\binom{13}{4} + \binom{26}{4}$$
$$= 2\frac{13!}{4!\,9!} + \frac{26!}{4!\,22!} = 13\cdot11\cdot10 + 26\cdot25\cdot23 = \boxed{16,380}$$

6. (6 points) In how many ways can you place 20 identical balls into five different boxes?

$$\underbrace{***}_{\text{Box 1}} \mid \underbrace{*****}_{\text{Box 2}} \mid \underbrace{*****}_{\text{Box 3}} \mid \underbrace{***}_{\text{Box 4}} \mid \underbrace{****}_{\text{Box 5}}$$

Answer $\binom{20+4}{4} = \binom{24}{4} = \frac{24 \cdot 23 \cdot 22 \cdot 21}{4 \cdot 3 \cdot 2}$

$= 23 \cdot 22 \cdot 21$

$= \boxed{10{,}626}$

7. (6 points) Suppose $a, b, c, d \in \mathbb{Z}$ and $n \in \mathbb{N}$.
Prove: If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

<u>Proof</u> (Direct) Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
This means $n \mid (a-b)$ and $n \mid (c-d)$ by def. of $\equiv \pmod{n}$.
By definition of divisibility, we then have
$a - b = nk$ and $c - d = n\ell$ for some $k, \ell \in \mathbb{Z}$.
Consequently $a = nk + b$ and $c = n\ell + d$.
Hence $ac = (nk+b)(n\ell+d) = n^2k\ell + nkd + bn\ell + bd$.
Therefore $ac - bd = n^2k\ell + nkd + bn\ell$
$= n(nk\ell + kd + b\ell)$
where $nk\ell + kd + b\ell \in \mathbb{Z}$.
From this the definition of divisibility gives
$n \mid (ac-bd)$, and thus $ac \equiv bd \pmod{n}$ ■

8. **(6 points) Prove:** If $n \in \mathbb{Z}$, then $4 \mid n^2$ or $4 \mid (n^2+3)$.

Proof (Direct) Suppose $n \in \mathbb{Z}$.

Case 1 Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$.
Therefore $n^2 = (2k)^2 = 4k^2$, meaning $\boxed{4 \mid n^2.}$

Case 2 Suppose n is odd. Then $n = 2k+1$ for $k \in \mathbb{Z}$
Note that $n^2+3 = (2k+1)^2+3 = 4k^2+4k+1+3$
$= 4k^2+4k+4 = 4(k^2+k+1)$ with $k^2+k+1 \in \mathbb{Z}$.
As $n^2+3 = 4(k^2+k+1)$, we have $\boxed{4 \mid (n^2+3)}$

Cases 1 and 2 above now show that $4 \mid n^2$
or $4 \mid (n^2+3)$. ▨

9. **(6 points) Prove:** If $n \in \mathbb{Z}$, then $4 \nmid (n^2-3)$.

Proof (Contradiction) Suppose for the sake of contradiction that $n \in \mathbb{Z}$ but $4 \mid (n^2-3)$. Then

$$n^2 - 3 = 4a \quad \text{for some } a \in \mathbb{Z},$$

$$n^2 = 4a+3$$

$$n^2 = 4a+2+1 = 2(2a+1)+1.$$

Therefore n^2 is odd, so n is odd, that is, $n = 2b+1$ for some $b \in \mathbb{Z}$. Now we have

$$n^2 - 3 = 4a$$

$$(2b+1)^2 - 3 = 4a$$

$$4b^2+4b+1-3 = 4a$$

$$4b^2+4b-2 = 4a$$

$$2b^2+2b-1 = 4a$$

$$2b^2+2b-4a = 1$$

$$2(b^2+b-2a) = 1$$

Therefore 1 is even, which is a contradiction ▨

10. (6 points) Suppose $a, b \in \mathbb{Z}$. Prove ab is odd if and only if both a and b are odd.

Proof ($\Rightarrow$) First we need to show that if ab is odd then both a and b are odd. We use contrapositive proof. Suppose that not both a and b are odd. Then at least one of them is even. Without loss of generality, say a is even, so $a = 2k$ for some $k \in \mathbb{Z}$. Then $ab = 2kb = 2(kb)$ with $kb \in \mathbb{Z}$, which means ab is even, so ab is not odd.

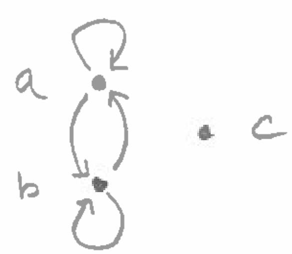
($\Leftarrow$) Now we need to prove that if a and b are both odd, then ab is odd. Let's use direct proof. Assume that both a and b are odd. Then $a = 2k+1$ and $b = 2l+1$ for $k, l \in \mathbb{Z}$. Now $ab = (2k+1)(2l+1) = 4kl + 2l + 2k + 1 = 2(2kl + l + k) + 1$. Because $2kl + l + k \in \mathbb{Z}$, this means ab is odd. $\blacksquare$

11. (6 points) Prove or disprove: If a relation R on a set A is both transitive and symmetric, then it is also reflexive.

This is FALSE. Here is a counterexample.

Let $A = \{a, b, c\}$

and $R = \{(a,a), (a,b), (b,a), (b,b)\}$



This is both transitive and symmetric, but it is not reflexive because $(c,c) \notin R$.

The questions on this page involve the function $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R} \times \mathbb{N}$ defined as $f((x,y)) = (3xy, y)$

12. (6 points) Is f is injective? Let's check. Suppose $f((a,b)) = f((c,d))$
Then $(3ab, b) = (3cd, d)$, which means $\boxed{3ab = 3cd}$

and $\boxed{b = d}$. Putting thes together gives $3ab = 3cb$ and hence $\boxed{a = c}$. From this, $(a,b) = (c,d)$ which proves $\boxed{f \text{ is injective.}}$

13. (6 points) Is f is surjective?

Given $(a,b) \in \mathbb{R} \times \mathbb{N}$, note that $\left(\frac{a}{3b}, b\right) \in \mathbb{R} \times \mathbb{N}$ and $f\left(\left(\frac{a}{3b}, b\right)\right) = \left(3\frac{a}{3b}b, b\right) = (a,b)$ so

$\boxed{f \text{ is surjective}}$

14. (6 points) Does the inverse function f^{-1} exist? If so, find it.

Because it's injective and surjective, f is bijective and Thus has an inverse

#13 above suggests that

$$\boxed{f^{-1}(x,y) = \left(\frac{x}{3y}, y\right)}$$

Check: $f^{-1}(f(x,y)) = f^{-1}((3xy, y))$

$$= \left(\frac{3xy}{3y}, y\right) = (x,y) \checkmark$$

15. (6 points) Use mathematical induction to prove $2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^n = 2^{n+1} - 2$ for every $n \in \mathbb{N}$.

Proof

① If $n=1$, this is $2^1 = 2^{1+1} - 2$, that is, $2 = 4-2$, and that's true!

② Now we need to show $2^1+2^2+2^3+\cdots+2^k = 2^{k+1}-2$ implies $2^1+2^2+2^3+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-2$.

We use direct proof. Suppose that $2^1+2^2+2^3+\cdots+2^k = 2^{k+1}-2$. Then

$$2^1+2^2+2^3+\cdots+2^k+2^{k+1} =$$

$$\left(2^1+2^2+2^3+\cdots+2^k\right)+2^{k+1} =$$

$$\left(2^{k+1}-2\right)+2^{k+1} =$$

$$2\cdot 2^{k+1}-2 = 2^1 2^{k+1} - 2$$

$$= 2^{k+1+1} - 2$$

$$= 2^{(k+1)+1} - 2.$$

So we've shown that

$$2^1+2^2+2^3+\cdots+2^{k+1} = 2^{(k+1)+1} - 2.$$

This completes the proof by induction.

16. (6 points) Prove: If $a, b \in \mathbb{Z}$, then $a^2 - 4b - 3 \neq 0$. [Contradiction may be easiest.]

Proof Suppose for the sake of contradiction that $a, b \in \mathbb{Z}$, but $a^2 - 4b - 3 = 0$. Then $a^2 = 4b + 3 = 4b + 2 + 1 = 2(2b+1) + 1$. So $a^2 = 2(2b+1) + 1$, where $2b+1 \in \mathbb{Z}$, and this means a^2 is odd, so consequently a is odd. Therefore $a = 2k+1$ for some $k \in \mathbb{Z}$.

Now plug $a = 2k+1$ into $a^2 - 4b - 3 = 0$ to get

$$(2k+1)^2 - 4b - 3 = 0$$

$$4k^2 + 4k + 1 - 4b - 3 = 0$$

$$4k^2 + 4k - 4b = 2$$

$$\frac{1}{2}(4k^2 + 4k + 4b) = \frac{1}{2} \cdot 2$$

$$2k^2 + 2k + 2b = 1$$

$$2(k^2 + k + b) = 1$$

Thus we have $1 = 2(k^2 + k + b)$, where $k^2 + k + b \in \mathbb{Z}$, which means 1 is even, a contradiction $\blacksquare$