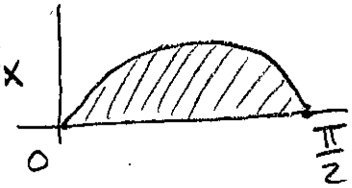
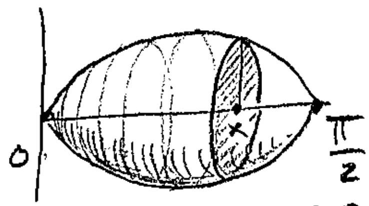
Name: Richard	Test 2 ♢ March 28, 2024	MATH 201 R. Hammack

1. The region under $y = \sin^2(x)\cos^{3/2}(x)$, and over the interval $[0, \pi/2]$ is rotated around the x-axis. Find the volume of the resulting solid.

$$V = \int_0^{\pi/2} A(x)\,dx = \int_0^{\pi/2} \pi\left(\sin^2(x)\cos^{3/2}(x)\right)^2 dx$$



$$= \pi\int_0^{\pi/2} \sin^4(x)\cos^3(x)\,dx$$



$$= \pi\int_0^{\pi/2} \sin^4(x)\cos^2(x)\cos(x)\,dx$$

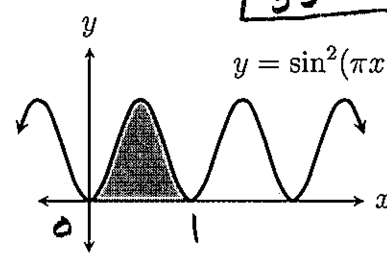
$$= \pi\int_0^{\pi/2} \sin^4(x)(1-\sin^2(x))\cos(x)\,dx$$

$u = \sin(x)$
$du = \cos(x)dx$

$$= \pi\int_0^1 u^4(1-u^2)\,du = \pi\int_0^1 u^4 - u^6\,du = \pi\left[\frac{u^5}{5} - \frac{u^7}{7}\right]_0^1 = \pi\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$= \boxed{\frac{2\pi}{35} \text{ cubic units}}$$

2. Find the area of the shaded region.



$$A = \int_0^1 \sin^2(\pi x)\,dx$$

$u = \pi x$
$du = \pi dx$
$\frac{1}{\pi}du = dx$

$$= \int_{\pi\cdot 0}^{\pi\cdot 1} \sin^2(u)\frac{1}{\pi}\,du$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin^2(u)\,du = \frac{1}{\pi}\left[\frac{u}{2} - \frac{\cos(u)\sin(u)}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\left(\frac{\pi}{2} - \frac{\cos\pi\,\sin\pi}{2}\right) - \left(\frac{0}{2} - \frac{\cos 0\,\sin 0}{2}\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - 0 - 0 + 0\right) = \boxed{\frac{1}{2} \text{ square unit}}$$

3. $\displaystyle\int \frac{dx}{1-\sin(\pi x)} = \int \frac{1}{1-\sin(\pi x)} \frac{1+\sin(\pi x)}{1+\sin(\pi x)}\, dx$

$$= \int \frac{1+\sin(\pi x)}{1-\sin^2(\pi x)}\, dx = \int \frac{1+\sin(\pi x)}{\cos^2(\pi x)}\, dx$$

$$= \int \frac{1}{\cos^2(\pi x)} + \frac{\sin(\pi x)}{\cos^2(\pi x)}\, dx = \int \sec^2(\pi x) + \frac{1}{\cos(\pi x)} \frac{\sin(\pi x)}{\cos(\pi x)}\, dx$$

$$= \int \sec^2(\pi x) + \sec(\pi x)\tan(\pi x)\, dx = \boxed{\frac{1}{\pi}\tan(\pi x) + \frac{1}{\pi}\sec(\pi x) + C}$$

4. $\displaystyle\int \cos^5(x)\, dx = \int \cos^4(x)\cos(x)\, dx = \int \left(\cos^2(x)\right)^2 \cos(x)\, dx$

$$= \int \left(1-\sin^2(x)\right)^2 \cos(x)\, dx = \int (1-u^2)^2\, du$$

$u = \sin x$
$du = \cos x\, dx$

$$= \int 1 - 2u^2 + u^4\, du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \boxed{\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C}$$

5. $\displaystyle\int \frac{x}{x^2+2x+1}\, dx = \int \frac{x}{(x+1)^2}\, dx = \int \frac{1}{x+1} - \frac{1}{(x+1)^2}\, dx$

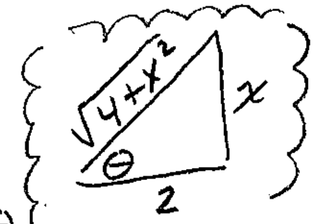
$$= \boxed{\ln|x+1| + \frac{1}{x+1} + C}$$

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x = A(x+1) + B$$

$$x = Ax + A + B$$

$$\Rightarrow \boxed{A=1} \quad A+B=0, \quad 1+B=0 \quad \boxed{B=-1}$$

6. $\displaystyle\int \frac{dx}{x\sqrt{4+x^2}} = \int \frac{2\sec^2\theta}{2\tan\theta\,\sqrt{4+(2\tan\theta)^2}}\,d\theta$



$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{\sec^2\theta}{\tan\theta\; 2\sqrt{1+\tan^2\theta}}\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\tan\theta\sqrt{\sec^2\theta}}\,d\theta = \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\,d\theta = \frac{1}{2}\int \frac{1}{\cos\theta}\,\frac{\cos\theta}{\sin\theta}\,d\theta$$

$$= \frac{1}{2}\int \csc\theta\,d\theta = -\frac{1}{2}\ln|\csc\theta + \cot\theta| + C$$

$$= \boxed{-\frac{1}{2}\ln\left|\frac{\sqrt{4+x^2}}{x} + \frac{2}{x}\right| + C}$$

7. $\displaystyle\int \frac{3x^3+2x^2+12x+9}{x^2+4}\,dx =$

$$= \int 3x + 2 + \frac{1}{x^2+4}\,dx = \boxed{\frac{3x^2}{2} + 2x + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

$$\begin{array}{r|l} & 3x + 2 \\ \hline x^2+4 & 3x^3 + 2x^2 + 12x + 9 \\ & 3x^3 + 12x \\ \hline & 2x^2 + 9 \\ & 2x^2 + 8 \\ \hline & 1 \end{array}$$

8. $\displaystyle\int x^3\sin(x^2)\,dx = \int x^2\sin(x^2)\,x\,dx = -\frac{x^2\cos x^2}{2} - \int -\frac{1}{2}\cos(x^2)\,2x\,dx$

$u = x^2 \quad dv = \sin(x^2)\,x\,dx$

$du = 2x\,dx \quad v = -\frac{1}{2}\cos(x^2)$

$$= \frac{-x^2\cos x^2}{2} + \int \cos(x^2)\,x\,dx$$

$$= \boxed{-\frac{x^2\cos x^2}{2} + \frac{\sin(x^2)}{2} + C}$$

9. $\displaystyle\int_1^\infty e^{1-x}\,dx = \lim_{b\to\infty}\int_1^b e^{1-x}\,dx = \lim_{b\to\infty}\Big[-e^{1-x}\Big]_1^b$

$$= \lim_{b\to\infty}\left(-e^{1-b}-\left(-e^{1-1}\right)\right) = \lim_{b\to\infty}\left(\frac{-1}{e^{b-1}}+e^0\right)$$

$$= 0 + e^0 = \boxed{1}$$

10. $\displaystyle\int_0^1 \ln(x)\,dx =$ (Note that $\ln(x)$ is not continuous on $[0,1]$!)

$$= \lim_{a\to0^+}\int_a^1 \ln(x)\,dx = \lim_{a\to0^+}\Big[x\ln(x) - x\Big]_a^1$$

$$= \lim_{a\to0^+}\Big(\big(1\ln(1)-1\big)-\big(a\ln(a)-a\big)\Big)$$

$$= \lim_{a\to0^+}\big(1\cdot0 - 1 - a\ln(a) + a\big)$$

$$= -1 - \lim_{a\to0^+} a\ln(a) = -1 - \lim_{a\to0^+}\frac{\ln(a)}{\frac{1}{a}}$$

form $\frac{\infty}{\infty}$

$$= -1 - \lim_{a\to0^+}\frac{\frac{1}{a}}{\frac{-1}{a^2}} = -1 - \lim_{a\to0^+} -\frac{a^2}{a}$$

$$= -1 - \lim_{a\to0^+}(-a) = -1 - 0 = \boxed{-1}$$