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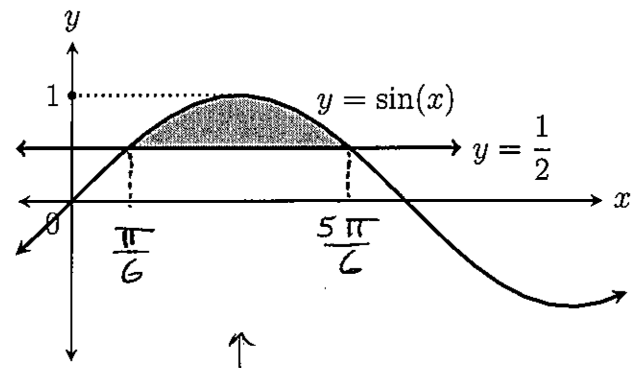
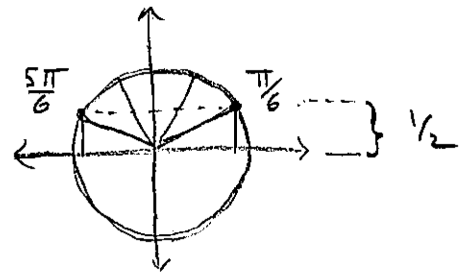
Test 1 ♣

MATH 201
February 13, 2024

1. Find the area of the shaded region.





$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin(x) - \tfrac{1}{2}\ dx$$

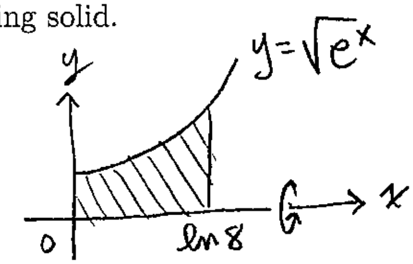
$$= \left[-\cos(x) - \frac{x}{2}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

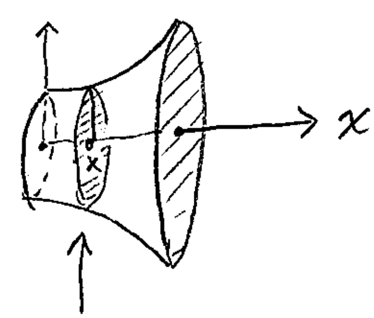
$$= \left(-\cos\left(\frac{5\pi}{6}\right) - \frac{5\pi/6}{2}\right) - \left(-\cos\left(\frac{\pi}{6}\right) - \frac{\pi/6}{2}\right)$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{5\pi}{12}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{12}\right)$$

$$= \sqrt{3} - \frac{5\pi}{12} + \frac{\pi}{12} = \sqrt{3} - \frac{4\pi}{12} = \boxed{\sqrt{3} - \frac{\pi}{3} \text{ square units}}$$

2. Consider the region bounded by $y = \sqrt{e^x}$, $y = 0$, $x = 0$ and $x = \ln(8)$.
This region is rotated around the x-axis. Find the volume of the resulting solid.





Volume by slicing

As noted on the right

$A(x) = \pi e^x$

$$V = \int_0^{\ln 8} A(x)\,dx = \int_0^{\ln 8} \pi e^x\,dx$$

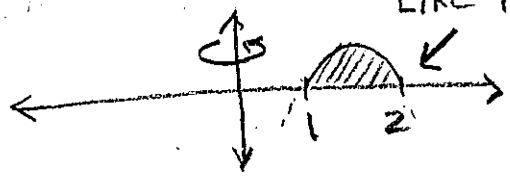
$$= \left[\pi e^x\right]_0^{\ln 8} = \pi\left(e^{\ln 8} - e^0\right)$$

$$= \pi(8-1) = \boxed{7\pi \text{ cubic units}}$$

$$A(x) = \pi r^2 = \pi \sqrt{e^x}^{\,2} = \pi e^x$$

3. The region contained between the x-axis and $\boxed{y = 3x - x^2 - 2}$ is rotated around the y-axis. Find the volume of the resulting solid.

Parabola, opens down. Like this



First, find x-intercepts:

$3x - x^2 - 2 = 0$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

So intercepts are 1 and 2.

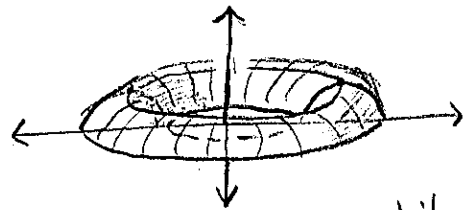
Volume by shells:

$$V = \int_1^2 2\pi x\, f(x)\, dx =$$

$$2\pi \int_1^2 x(3x - x^2 - 2)\, dx =$$

$$2\pi \int_1^2 3x^2 - x^3 - 2x\, dx = 2\pi \left[x^3 - \frac{x^4}{4} - x^2\right]_1^2 =$$

$$2\pi\left(\left(2^3 - \frac{2^4}{4} - 2^2\right) - \left(1^3 - \frac{1^4}{4} - 1^2\right)\right) = 2\pi\left((8-4-4) - \left(\frac{-1}{4}\right)\right) = \boxed{\frac{\pi}{2} \text{ cubic units}}$$



Rotated, looks like a bundt cake!

4. Find the arc length of the curve $y = \dfrac{\sqrt{x^2+2}^{\,3}}{3}$ from $x = 0$ to $x = 1$.

$y = \frac{1}{3}(x^2+2)^{3/2}$

$y' = \frac{1}{2}(x^2+2)^{1/2} \cdot 2x$

$y' = \sqrt{x^2+2}\, x$

$$L = \int_0^1 \sqrt{1 + (f'(x))}\, dx$$

$$= \int_0^1 \sqrt{1 + \left(\sqrt{x^2+2}\, x\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + (x^2+2)x^2}\, dx = \int_0^1 \sqrt{1 + x^4 + 2x^2}\, dx$$

$$= \int_0^1 \sqrt{x^4 + 2x^2 + 1}\, dx = \int_0^1 \sqrt{(x^2+1)^2}\, dx = \int_0^1 x^2 + 1\, dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 = \frac{1}{3} + 1 = \boxed{\frac{4}{3} \text{ units}}$$

5. The graph of $y = x^3$ for $0 \le x \le 1$ is rotated around the x-axis. Find the area of the resulting surface.

$$\int_0^1 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = \int_0^1 2\pi x^3 \sqrt{1+(3x^2)^2}\,dx$$

$$= 2\pi \int_0^1 \sqrt{1+9x^4}\; x^3\,dx \qquad \begin{aligned} u &= 1+9x^4 \\ du &= 36x^3\,dx \\ x^3\,dx &= \tfrac{1}{36}\,du \end{aligned}$$

$$= 2\pi \int_{1+9\cdot 0^4}^{1+9\cdot 1^4} \sqrt{u}\;\frac{1}{36}\,du$$

$$= \frac{\pi}{18}\int_1^{10} u^{1/2}\,du = \frac{\pi}{18}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^{10} = \frac{\pi}{18}\left[\frac{2\sqrt{u}^{\,3}}{3}\right]_1^{10}$$

$$= \frac{\pi}{27}\left(\sqrt{10}^{\,3}-\sqrt{1}^{\,3}\right) = \boxed{\frac{\pi}{27}\left(10\sqrt{10}-1\right) \text{ square units}}$$

6. A variable force moves an object from 0 to 5 on the number line (units in meters). At any point x between 0 and 5, the force is $\dfrac{2x}{x^2+1}$ Newtons. Find the work done in moving the object from 0 to 5.

$$W = \int_0^5 \frac{2x}{x^2+1}\,dx \qquad \begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}$$

$$= \int_{0^2+1}^{5^2+1} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{26}$$

$$= \ln|26| - \ln|1| = \ln(26) - 0$$

$$= \boxed{\ln(26)\ \text{J}}$$