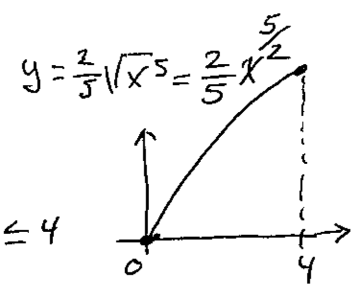
§ 8.8 Numerical Integration

It is still a fact that many (most?) integrals cannot be evaluated by any known means. This section concerns approximating such definite integrals by certain numeric methods.



Motivational Example

Find the arclength of $y = \frac{2}{5}\sqrt{x}^{\,5}$ for $0 \le x \le 4$

$$L = \int_0^4 \sqrt{1 + (f'(x))^2}\,dx = \int_0^4 \sqrt{1 + \left(x^{3/2}\right)^2}\,dx$$

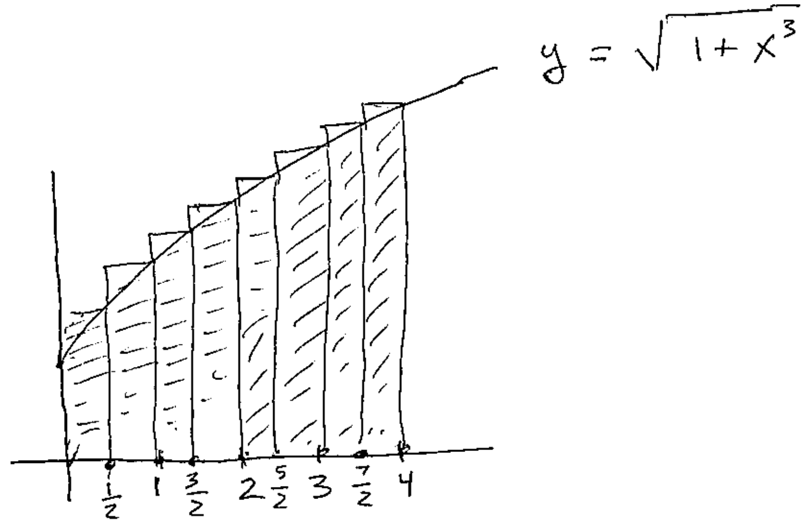
$$= \int_0^4 \sqrt{1 + x^3}\,dx \longleftarrow \text{impossible to evaluate!}$$

Since the fundamental theorem of calculus fails us (because we can't find an antiderivative), we can at least approximate this integral with rectangles:

$n = 8$

$\Delta x = \frac{4-0}{8} = \frac{1}{2}$

$x_0 = 0$
$x_1 = \frac{1}{2}$
$x_2 = 1$
$x_3 = \frac{3}{2}$
$\vdots$
$x_n = 4$

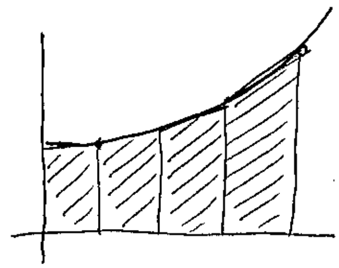


$$\int_0^4 \sqrt{1+x^2}\,dx \approx \sum_{k=1}^{n} \sqrt{1 + (x_k)^3}\,\Delta x = \sum_{k=1}^{n} \sqrt{1 + \left(\tfrac{k}{n}\right)^3}\,\frac{1}{2}$$

$$= \left(\sqrt{1+\left(\tfrac{1}{2}\right)^3} + \sqrt{1+1^3} + \sqrt{1+\left(\tfrac{3}{2}\right)^3} + \cdots + \sqrt{1+4^3}\right)\frac{1}{2}$$

$=$ etc. (not much fun.)

A better approach would be to approximate with <u>trapezoids</u> instead of rectangles



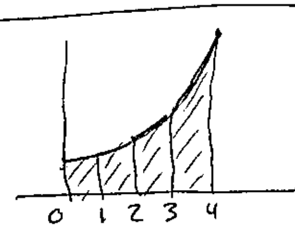
Here's how it works out:

<u>Trapezoid Rule</u> For a positive (large!) integer n,

$$\int_a^b f(x)\,dx \approx \left(\tfrac{1}{2} f(x_0) + f(x_1) + f(x_2) + f(x_3) + \dots + f(x_{n-1}) + \tfrac{1}{2} x_n\right) \Delta x$$

where $\Delta x = \dfrac{b-a}{n}$, $x_0 = a$ $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, $\dots$, $x_n = b$

<u>Example</u> Let $n=4$ $\left\{ \begin{array}{lll} \Delta x = \frac{4-0}{4} = 1 & & \\ x_0 = 0 & x_2 = 2 & x_4 = 4 \\ x_1 = 1 & x_3 = 3 & \end{array} \right.$



$$\int_0^4 \sqrt{1+x^3}\,dx \approx \tfrac{1}{2}\sqrt{1+0^3} + \sqrt{1+1^3} + \sqrt{1+2^3} + \sqrt{1+3^3} + \tfrac{1}{2}\sqrt{1+4^3}$$

$$= \tfrac{1}{2} + \sqrt{2} + \sqrt{9} + \sqrt{28} + \tfrac{1}{2}\sqrt{65}$$

$$= 14.23684$$

Section 8.4 describes other methods, such as Simpson's Rule, which approximates with parabolas. Also presented are bounds on errors for different values of n.

But all This is best left for a course on <u>numerical</u> <u>methods</u>.

The takeaway for us in Calculus II is that when the fundamental theorem of calculus cannot be applied, then there exist methods to overcome this.