§ 8.5 Rational Functions

Recall A rational function is one of form $f(x) = \dfrac{p(x)}{q(x)}$ ← polynomial / ← polynomial

A rational function is proper if degree $(p(x))$ < degree $(q(x))$

You can integrate some rational functions with ln tan or power rule.

- $\displaystyle\int \frac{2x+3}{x^2+3x+2}\,dx = \ln|x^2+3x+2| + C$
- $\displaystyle\int \frac{1}{5+x^2}\,dx = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$
- $\displaystyle\int \frac{2}{x+1}\,dx = 2\ln|x+1| + C$
- $\displaystyle\int \frac{2x}{(1+x^2)^3} = \frac{-1}{2(1+x^2)^2} + C$

If a rational function is not proper, try Long division.

- $\displaystyle\int \frac{x^4+x-1}{x^2-1}\,dx = \int x^2+1+\frac{x}{x^2-1}\,dx$

$$\begin{array}{r|l} & \quad x^2 \qquad\quad +1 \\ \hline x^2-1 & x^4+0x^3+0x^2+x-1 \\ & x^4 \qquad\quad -x^2 \\ \hline & \qquad\qquad x^2+x-1 \\ & \qquad\qquad x^2 \qquad -1 \\ \hline & \qquad\qquad\quad x \end{array}$$

$= \dfrac{x^3}{3} + x + \dfrac{1}{2}\ln|x^2-1| + C$

But some proper rational functions don't fit these patterns

Ex $\displaystyle\int \frac{5x-3}{x^2-2x-3}\,dx = ?$

Today we learn how to do integrals like this. The key is adding fractions

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2}{x-1}\frac{x-3}{x-3} + \frac{3}{x-3}\frac{x+1}{x+1} = \frac{2x-6+3x+3}{(x-3)(x+1)} = \frac{5x-3}{x^2-2x-3}$$

partial fraction decomposition of the rational function

rational function

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1} + \frac{3}{x-3}\,dx = \boxed{2\ln|x+1| + 3\ln|x-3| + C}$$

But how would we know $\dfrac{5x-3}{x^2-2x-3} \stackrel{?}{=} \dfrac{2}{x+1} + \dfrac{3}{x-3}$?

$$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} \Rightarrow \boxed{5x-3 = A(x-3) + B(x+1)}$$

Put $x=3$ Get $5\cdot 3-3 = B(3+1)$ $\quad 12 = 4B$ $\quad \boxed{B=3}$

Put $x=-1$ Get $5(-1)-3 = A(-1-3)$ $\quad -8 = -4A$ $\quad \boxed{A=2}$

$\therefore \dfrac{5x-3}{(x+1)(x-3)} = \dfrac{2}{x+1} + \dfrac{3}{x-3}$

Ex $\displaystyle\int \frac{4x+11}{x^2+x-2}\,dx = \int \frac{5}{x-1} - \frac{1}{x+2}\,dx = \boxed{5\ln|x-1| - \ln|x+2| + C}$

$$\frac{4x+11}{x^2+x-2} = \frac{4x+11}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$\Rightarrow \quad 4x+11 = A(x+2) + B(x-1)$

$x=-2: \quad 3 = -3B \Rightarrow \boxed{B=-1}$

$x=1: \quad 15 = 3A \Rightarrow \boxed{A=5}$

Alternative: $4x+11 = (A+B)x + 2A - B$

$A+B = 4$

$2A - B = 11$

$3A = 15$

$A = 5$ etc.

Ex $\displaystyle\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}\,dx = \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$

$= \ln|x-1| - \ln|(x-2)^5| + \ln|(x-3)^5| + C$

$= \boxed{\ln\left|\frac{(x-1)(x-3)^5}{(x-2)^5}\right| + C}$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$x=1: \quad 2 = A(1-2)(1-3) \quad 2 = 2A \quad \boxed{A=1}$

$x=2: \quad 5 = B(2-1)(2-3) \quad 5 = -B \quad \boxed{B=-5}$

$x=3: \quad 10 = C(3-1)(3-2) \quad 10 = 2C \quad \boxed{C=5}$

Ex $\displaystyle\int \frac{10\cos x}{\sin^2 x + \sin(x) - 6}\,dx = \int \frac{10}{u^2+u-6}\,du = \int \frac{2}{u-2} - \frac{2}{u+3}\,du$

$= 2\ln|u-2| - 2\ln|u+3| + C = \ln\left|\frac{(u-2)^2}{(u+3)^2}\right| + C$

$= \boxed{\ln\left|\frac{\sin x - 2}{\sin x + 3}\right|^2 + C}$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$\frac{10}{u^2+u-6} = \frac{10}{(u-2)(u+3)} = \frac{A}{u-2} + \frac{B}{u+3} \Longrightarrow 10 = A(u+3) + B(u-2)$$

$u=2: \quad 10 = A(2+3) \quad A = 2$

$u=-3: \quad 10 = B(-3-2) \quad B = -2$

Ex $\displaystyle\int \frac{2x+3}{(x+1)^2}\,dx = \int \frac{2}{x+1} + \frac{1}{(x+1)^2}\,dx = \boxed{2\ln|x+1| - \frac{1}{x+1} + C}$

$$\frac{2x+3}{(x+1)(x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Longrightarrow 2x+3 = A(x+1) + B$$

$= Ax + (A+B)$

$\uparrow \quad\quad \uparrow$

$2 \quad\quad 1$

$A = 2$

$B = 1$

The Way It Works (By Example)

① Linear Factors in denominator

$$\frac{p(x)}{(x-2)(x-3)^3} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3}$$

② Irreducible quadratic factor in denominator

$$\frac{p(x)}{x^2+x+5} = \frac{Ax+B}{x^2+x+5} = \frac{Ax}{x^2+x+5} + \frac{B}{x^2+x+5}$$

irreducible i.e. can't be factored

Ex

$$\frac{2x^2+2}{(x-2)^3(x^2+x+5)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+x+5} + \frac{Fx+G}{(x^2+x+5)^2}$$

Ex $\displaystyle\int \frac{2x^3+10x}{(x^2+1)^2}\,dx = \int\left(\frac{2x}{x^2+1} + \frac{8x}{(x^2+1)^2}\right)dx = \boxed{\ln|x^2+1| + 4\frac{1}{x^2+1} + C}$

$$\frac{2x^3+10x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{2x}{x^2+1} + \frac{8x}{(x^2+1)^2}$$

$$2x^3+10x = (Ax+B)(x^2+1) + Cx + D$$
$$2x^3+10x = Ax^3 + Ax + Bx^2 + B + Cx + D$$
$$2x^3+10x = Ax^3 + Bx^2 + (A+C)x + B+D$$

$A = 2$

$B = 0$

$A+C = 10 \qquad C = 8$

$B+D = 0 \qquad D = 0$

Ex $\displaystyle\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx = \int\left(\frac{5}{x+1} + \frac{1}{(x+1)^2} - \frac{3}{x-5}\right)dx = \boxed{5\ln|x+1| - \frac{1}{x+1} - 3\ln|x-5| + C}$

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5}$$

$$2x^2-25x-33 = A(x+1)(x-5) + B(x-5) + C(x+1)^2 = (A+C)x^2 + \cdots$$

$x=-1$: $-6 = B(-1-5)$ $\boxed{B=1}$ $\qquad A+C = 2 \Rightarrow \boxed{A=5}$

$x=5$: $-108 = C(5+1)^2$ $\boxed{C=-3}$

Ex $\displaystyle\int \frac{x^3+x^2+3x+1}{(x^2+1)^2}\,dx = \int\left(\frac{x+1}{x^2+1} + \frac{2x}{(x^2+1)^2}\right)dx = \int\left(\frac{x}{x^2+1} + \frac{1}{x^2+1} + \frac{2x}{(x^2+1)^2}\right)$

$$= \frac{1}{2}\ln|x^2+1| + \tan^{-1}(x) + \frac{-1}{x^2+1} + C$$

$$\frac{x^3+x^2+3x+1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$x^3+x^2+3x+1 = (Ax+B)(x^2+1) + Cx + D$$
$$x^3+x^2+3x+1 = Ax^3 + Bx^2 + (A+C)x + (D+B)$$

$\boxed{A=1}$ $\quad \boxed{B=1}$ $\quad \boxed{D=0}$

$A+C = 3 \Rightarrow \boxed{C=2}$