

Section 8.3 Trigonometric Integrals

Our goal in this section is to learn how to evaluate integrals involving trig functions.

Basic Definitions

	$\sin(\alpha) = \frac{\text{OPP}}{\text{HYP}}$	$\tan(\alpha) = \frac{\text{OPP}}{\text{ADJ}}$	$\sec(\alpha) = \frac{\text{HYP}}{\text{ADJ}}$
	$\cos(\alpha) = \frac{\text{ADJ}}{\text{HYP}}$	$\cot(\alpha) = \frac{\text{ADJ}}{\text{OPP}}$	$\csc(\alpha) = \frac{\text{HYP}}{\text{OPP}}$

Basic Trig Triangles and Identities

<p>$\sin^2(\alpha) + \cos^2(\alpha) = 1$</p>	<p>$\tan^2(\alpha) + 1 = \sec^2(\alpha)$</p>	<p>$\cot^2(\alpha) + 1 = \csc^2(\alpha)$</p>
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Addition and Double-Angle Identities

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$	$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$	$\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$
$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$	$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$	$\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$

Basic Trig Integrals

$$\int \sin(u) du = -\cos(u) + C$$

$$\int \cos(u) du = \sin(u) + C$$

$$\int \tan(u) du = \ln|\sec(u)| + C$$

$$\int \cot(u) du = \ln|\sin(u)| + C$$

$$\int \sec(u) du = \ln|\sec(u) + \tan(u)| + C$$

$$\int \csc(u) du = -\ln|\csc(u) - \cot(u)| + C$$

$$\int \sec(u) \tan(u) du = \sec(u) + C$$

$$\int \csc(u) \cot(u) du = -\csc(u) + C$$

$$\int \sin^2(u) du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$$

$$\int \cos^2(u) du = \frac{u}{2} + \frac{\sin(2u)}{4} + C$$

$$\int \tan^2(u) du = \tan(u) - u + C$$

$$\int \cot^2(u) du = -\cot(u) - u + C$$

$$\int \sec^2(u) du = \tan(u) + C$$

$$\int \csc^2(u) du = -\cot(u) + C$$

Derived Below

- $\int \tan^2(u) du = \int \sec^2(u) - 1 du = \tan(u) - u + C$
- $\int \cot^2(u) du = \int \csc^2(u) - 1 du = -\cot(u) - u + C$
- $\int \sin^2(u) du = \int \frac{1}{2}(1 - \cos(2u)) du = \frac{u}{2} - \frac{\sin(2u)}{4} + C$
- $\int \cos^2(u) du = \int \frac{1}{2}(1 + \cos(2u)) du = \frac{u}{2} + \frac{\sin(2u)}{4} + C$

Ex $\int \frac{\cot^2(\ln(x))}{x} dx =$

$u = \ln(x)$
 $du = \frac{1}{x} dx$

$= \int \cot^2(u) du$

$= -\cot(u) - u + C$

$= -\cot(\ln(u)) - \ln(u) + C$

But what should we make of $\int \sin^5 x \, dx$? Familiar substitutions don't work. We have to be clever and use some identities.

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx$$

$$\begin{aligned} & \text{Let } u = \cos x \\ & du = -\sin x \, dx \end{aligned}$$

$$= -\int (1 - u^2)^2 \, du = -\int (1 - 2u^2 + u^4) \, du = -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C$$

$$= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}$$

That wasn't so bad, but note we would be in trouble if the power 5 had been even. Also if the power was at all large this method wouldn't have been adequate. For general integrals of form $\int \cos^n x \, dx$ and $\int \sin^n x \, dx$ we use

REDUCTION FORMULAS

$$\int \sin^n(x) \, dx = -\frac{1}{n} \sin^{n-1}(x) \cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x) \, dx$$

$$\int \cos^n(x) \, dx = \frac{1}{n} \cos^{n-1}(x) \sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x) \, dx$$

(These are derived using integration by parts - see that section)

Ex $\int \sin^2(x) \, dx = -\frac{1}{2} \sin(x) \cos(x) + \frac{1}{2} \int dx = \frac{1}{2}x - \frac{1}{2} \sin(x) \cos(x)$

Ex

$$\int \cos^4 x \, dx = \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, dx$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \left(\frac{1}{2}x + \frac{1}{4} \sin 2x \right) + C$$

$$= \frac{1}{4} \cos^3 x \sin x + \frac{3}{8}x + \frac{3}{16} \sin 2x + C$$

Check: $\frac{d}{dx} \left[\frac{1}{4} \cos^3 x \sin x + \frac{3}{8}x + \frac{3}{16} \sin 2x \right]$

$$= \frac{1}{4} (3 \cos^2 x \sin^2 x + \cos^3 x \cos x) + \frac{3}{8} + \frac{3}{8} \cos(2x) \cdot 2$$

$$= \frac{3}{4} \cos^2 x \sin^2 x + \frac{1}{4} \cos^4 x + \frac{3}{8} + \frac{3}{8} \cos(2x)$$

$$= \frac{3}{4} \cos^2 x \sin^2 x + \frac{1}{4} \cos^4 x + \frac{3}{8} + \frac{3}{8} \cos^2 x - \frac{3}{8} \sin^2 x$$

$$= \frac{3}{4} \cos^2 x (1 - \cos^2 x) + \frac{1}{4} \cos^4 x + \frac{3}{8} + \frac{3}{8} \cos^2 x - \frac{3}{8} (1 - \cos^2 x) = \cos^4 x$$

$$\begin{aligned}
 \underline{\text{Ex}} \quad \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
 &= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \\
 &= \int (1 - u^2) u^2 (-1) du \\
 &= \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + C \\
 &= \boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}
 \end{aligned}$$

$$\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$$

$$\begin{aligned}
 \underline{\text{Ex}} \quad \int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x (\cos^2 x)^2 \cos x \, dx \\
 &= \int \sin^2 x (1 - \sin^2 x)^2 \cos x \, dx \\
 &= \int u^2 (1 - u^2)^2 du = \int u^2 (1 - 2u^2 + u^4) du \\
 &= \int (u^2 - 2u^4 + u^6) du = \frac{u^3}{3} - 2 \frac{u^5}{5} + \frac{u^7}{7} + C \\
 &= \boxed{\frac{\sin^3 x}{3} - 2 \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C}
 \end{aligned}$$

$$\begin{cases} u = \sin x \\ du = \cos x \, dx \end{cases}$$

$$\begin{aligned}
 \underline{\text{Ex}} \quad \int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{2}(1 - \cos 2x) \frac{1}{2}(1 + \cos 2x) \, dx \\
 &= \frac{1}{4} \int (1 - \cos^2 2x) \, dx = \frac{1}{4} \int \sin^2 2x \, dx \\
 &= \frac{1}{8} \int \sin^2(2x) \, 2 \, dx = \frac{1}{8} \int \sin^2(u) \, du \\
 &= \frac{1}{8} \left(\frac{1}{2} u - \frac{1}{2} \sin(u) \cos(u) \right) + C \\
 &= \frac{1}{8} \left(x - \frac{1}{2} \sin(2x) \cos(2x) \right) + C \\
 &= \frac{x}{8} - \frac{1}{16} \sin(2x) \cos(2x) + C
 \end{aligned}$$

$$\begin{cases} u = 2x \\ du = 2 \, dx \end{cases}$$

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \sec x \tan x \, dx$$

$$= \int (\sec^2(x) - 1) \sec^2 x (\sec x \tan x) \, dx$$

$$= \int (\sec^4 x - \sec^2 x) (\sec x \tan x) \, dx$$

$$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

$$u = \sec x$$

$$du = \sec x \tan x \, dx$$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^3(x) (\tan^2(x) + 1) \sec^2(x) \, dx$$

$$= \int (\tan^5(x) + \tan^3(x)) \sec^2(x) \, dx$$

$$= \int (u^5 + u^3) \, du = \frac{u^6}{6} + \frac{u^4}{4} + C$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

What about
 $\int \tan^2(x) \sec^3(x) \, dx$?
 See next page.

$$\int \sin(\pi x) \cos(3\pi x) \, dx = \int \frac{1}{2} [\sin(\pi x - 3\pi x) + \sin(\pi x + 3\pi x)] \, dx$$

$$= \frac{1}{2} \int (\sin(-2\pi x) + \sin(4\pi x)) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{-2\pi} \cos(-2\pi x) - \frac{1}{4\pi} \cos(4\pi x) \right) + C$$

$$= \frac{-1}{4\pi} \cos(-2\pi x) + \frac{1}{8\pi} \cos(4\pi x) + C$$

REDUCTION FORMULAS

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Ex $\int \tan^2 x \sec x \, dx$

$$= \int (\sec^2 x - 1) \sec x \, dx$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx - \int \sec x \, dx$$

$$= \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x| - \frac{1}{2} \int \sec x \, dx$$

$$= \frac{\sec x \tan x}{2} - \frac{1}{2} \ln |\sec x + \tan x| + C$$

Ex $\int \tan^2(x) \sec^3(x) \, dx = \int (\sec^2(x) - 1) \sec^3(x) \, dx$

$$= \int \sec^5(x) \, dx - \int \sec^3(x) \, dx = \text{Now use reduction formulas!}$$

= next page

$$= \int \sec^5(x) dx - \int \sec^3(x) dx$$

$$= \frac{\sec^3(x) \tan(x)}{4} + \frac{5-2}{5-1} \int \sec^3(x) dx - \int \sec^3(x) dx$$

$$= \frac{\sec^3(x) \tan(x)}{4} - \frac{1}{4} \int \sec^3(x) dx$$

$$= \frac{\sec^3(x) \tan(x)}{4} - \frac{1}{4} \left(\frac{\sec(x) \tan(x)}{2} + \frac{3-2}{3-1} \int \sec(x) dx \right)$$

$$= \left[\frac{\sec^3(x) \tan(x)}{4} - \frac{\sec(x) \tan(x)}{8} - \frac{1}{8} \ln|\sec(x) + \tan(x)| \right] + C$$