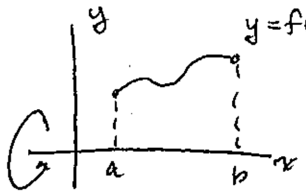
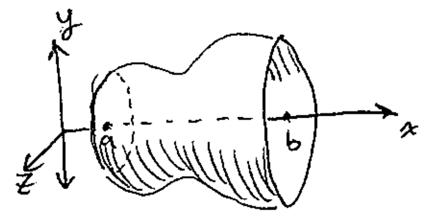
Section 6.6 Area of a surface of revolution

Definite integrals can also be used to find the area of surfaces in 3-D space, and this section gives us a taste of that. We will examine just one kind of surface, the so-called surface of revolution.

Basic problem

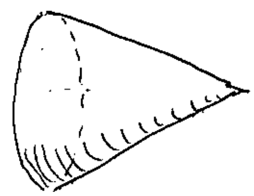


rotate around x-axis ....

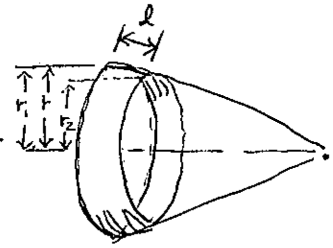


Now, what is the surface area of this shape?

To get a grip on the surface area, we recall two related shapes from geometry:



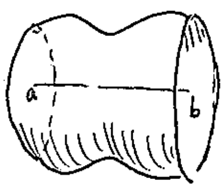
cone

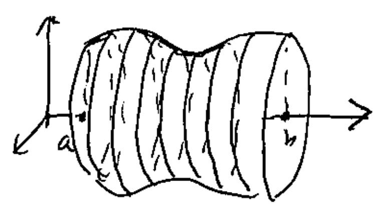


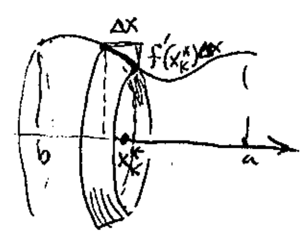
frustum of a cone

From Geometry
Surface area of frustum
$A = 2\pi r l$
(where $r = \frac{r_1 + r_2}{2}$)

Now back to our surface... Approximate it with n frustums.





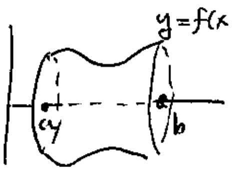


$$A \approx \sum_{k=1}^{n} \begin{pmatrix}\text{area of}\\ \text{frustum}\\ \#\, k\end{pmatrix} = \sum_{k=1}^{n} 2\pi r_k l_k = \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{\Delta x^2 + (f'(x_k^*)\Delta x)^2}$$

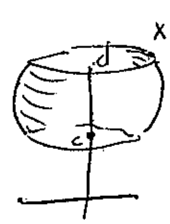
$$= \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{1 + (f'(x_k^*))^2}\,\Delta x$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi f(x_k^*) \sqrt{1 + (f'(x_k^*))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Conclusions

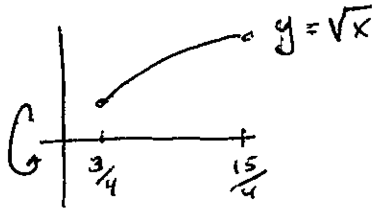


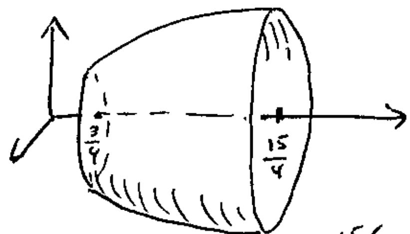
$$A = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$



$$A = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy$$

As you can probably imagine, the integrals coming out of these formulas are <u>tough</u> to evaluate unless the integrand is just so. But let's do a contrived example that <u>does</u> work out.





$$S = \int_{3/4}^{15/4} 2\pi \sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = \int_{3/4}^{15/4} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx$$

$$= 2\pi \int_{3/4}^{15/4} \sqrt{x+\tfrac{1}{4}}\,dx = 2\pi \int_{\frac{3}{4}+\frac{1}{4}}^{\frac{15}{4}+\frac{1}{4}} \sqrt{u}\,du = 2\pi \int_1^4 \sqrt{u}\,du$$

$$= 2\pi\left[\frac{2\sqrt{u}^{\,3}}{3}\right]_1^4 = 2\pi\left[\frac{2\sqrt{4}^{\,3}}{3} - \frac{2\sqrt{1}^{\,3}}{3}\right] = 2\pi\left(\frac{16}{3}-\frac{2}{3}\right) =$$

$$2\pi\left(\frac{14}{3}\right) = \boxed{\frac{28\pi}{3} \text{ square units}}$$