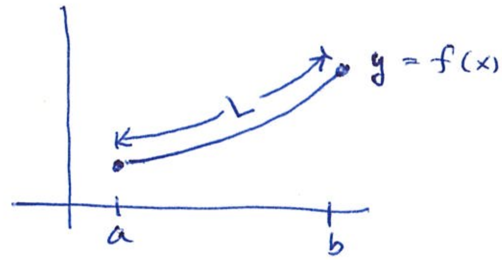
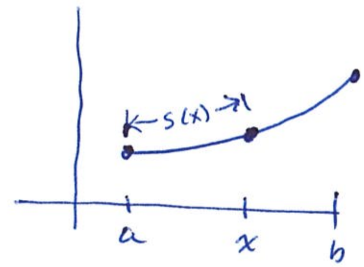
Section 6.5 Lengths of Curves

Basic Question: what is the length L of the curve $y = f(x)$ between $x = a$ and $x = b$?



We will develop a formula for L, then work out some examples.

To begin, let $s(x)$ = length from $(a, f(a))$ to $(x, f(x))$.
Then $s(a) = 0$ and $s(b) = L$



$$L = s(b) = s(b) - s(a) = \int_a^b s'(x)\,dx$$

Therefore, if we can find an expression for $s'(x)$, we will have a formula for L.

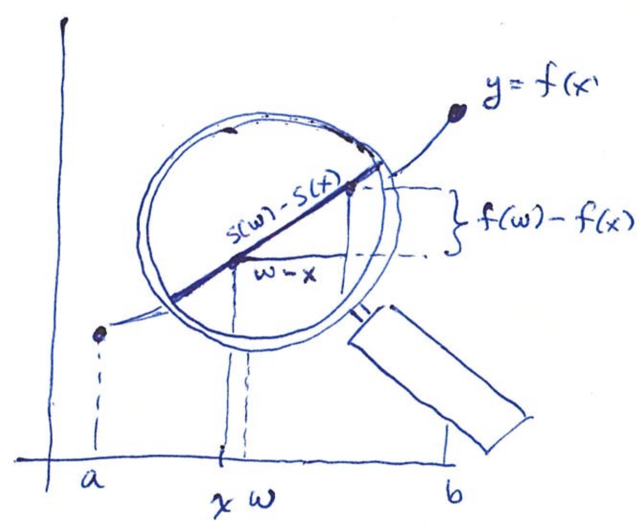
$$s'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

$$= \lim_{w \to x} \frac{\sqrt{(w-x)^2 + (f(w) - f(x))^2}}{w - x}$$

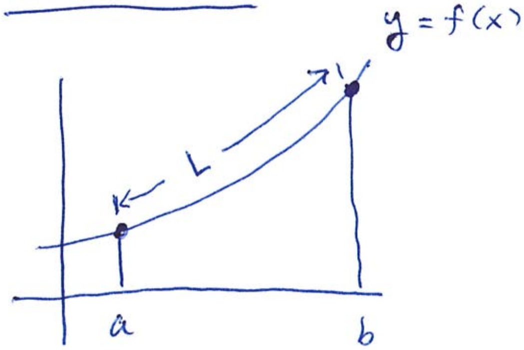
$$= \lim_{w \to x} \sqrt{\frac{(w-x)^2}{(w-x)^2} + \left(\frac{f(w) - f(x)}{w - x}\right)^2}$$

$$= \lim_{w \to x} \sqrt{1 + \left(\frac{f(w) - f(x)}{w - x}\right)^2}$$

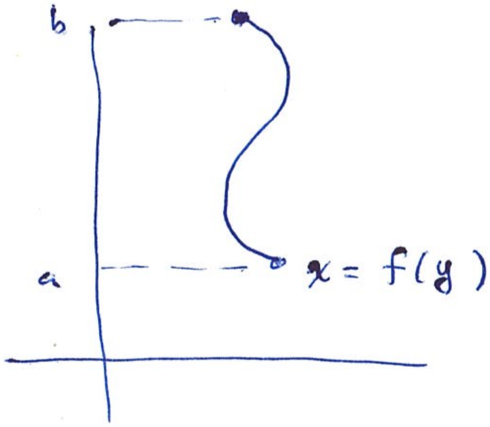
$$= \sqrt{1 + (f'(x))^2}\,dx$$



Conclusion

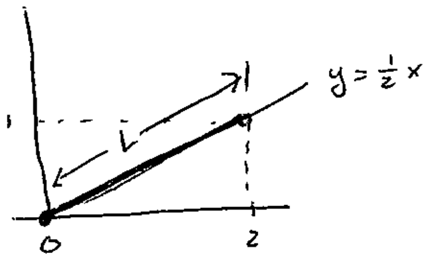


$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$



$$L = \int_a^b \sqrt{1 + (f'(y))^2}\, dy.$$

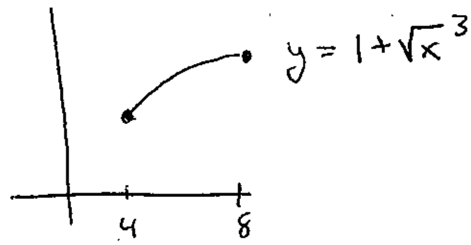
Ex For starters, let's test our formula on an example whose answer is easily computable using geometry.



By Pythagorean Theorem: $L = \sqrt{2^2+1^2} = \boxed{\sqrt{5}}$ units.

By Formula: $L = \int_0^2 \sqrt{1^2+(f'(x))^2}\,dx = \int_0^2 \sqrt{1+(1/2)^2}\,dx$

$= \int_0^2 \sqrt{5/4}\,dx = \left[\frac{\sqrt{5}}{2}x\right]_0^2 = \frac{\sqrt{5}}{2}2 - \frac{\sqrt{5}}{2}0 = \boxed{\sqrt{5}}$

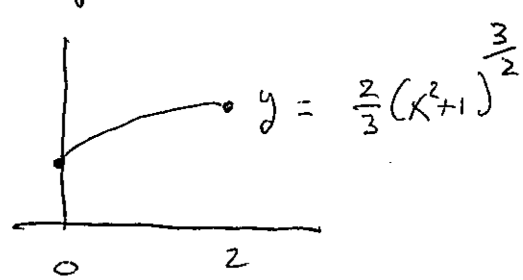


Ex $L = \int_4^8 \sqrt{1+(f'(x))^2}\,dx = \int_4^8 \sqrt{1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$

$= \int_4^8 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int_4^8 \sqrt{1+\frac{9}{4}x}\;\frac{9}{4}\,dx$

$= \frac{4}{9}\int_{1+\frac{9}{9}\cdot 4}^{1+\frac{9}{4}\cdot 8} \sqrt{u}\,du = \frac{4}{9}\int_{10}^{19} u^{\frac{1}{2}}\,du = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{10}^{19} = \frac{8}{27}\left[\sqrt{u}^3\right]_{10}^{19} = \frac{8}{27}\left(\sqrt{19}^3 - \sqrt{10}^3\right)$

As a general rule, integrals coming from $\int_a^b \sqrt{1+f'(x)^2}\,dx$ are very hard to evaluate, and we are going to stick with very special curves that make the math work out. In the real world, you often have to use the limit definition of a definite integral and settle for an approximation. Heres' another example.



$L = \int_0^2 \sqrt{1+\left((x^2+1)^{\frac{1}{2}}2x\right)^2}\,dx$

$= \int_0^2 \sqrt{1+(x^2+1)4x^2}\,dx$

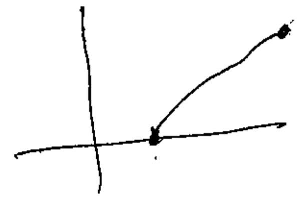
$= \int_0^2 \sqrt{4x^4+4x^2+1}\,dx$

$= \int_0^2 \sqrt{(2x^2+1)^2}\,dx = \int_0^2 (2x^2+1)\,dx = \left[2\frac{x^3}{3}+x\right]_0^2$

$= 2\cdot\frac{2^3}{3} + 2 = \frac{16}{3} + \frac{6}{3} = \frac{22}{3}$ units.

Ex Find the arc length of the curve

$$y = f(x) = \int_1^x \sqrt{2t + t^2}\, dt$$



between $x = 1$ and $x = 2$.

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$f'(x) = \sqrt{2x + x^2}$

Solution

$$L = \int_1^2 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \sqrt{2x + x^2}^{\,2}}\, dx$$

$$= \int_1^2 \sqrt{1 + 2x + x^2}\, dx$$

$$= \int_1^2 \sqrt{x^2 + 2x + 1}\, dx$$

$$= \int_1^2 \sqrt{(x+1)^2}\, dx = \int_1^2 (x+1)\, dx$$

$$= \left[\frac{x^2}{2} + x\right]_1^2 = \left(\frac{2^2}{2} + 2\right) - \left(\frac{1^2}{2} + 1\right) = 4 - \frac{3}{2} = \boxed{\frac{5}{2} \text{ units}}$$