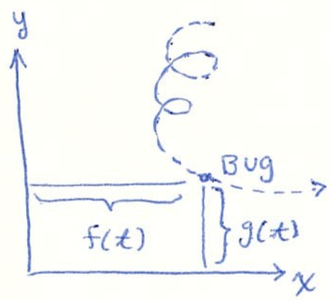
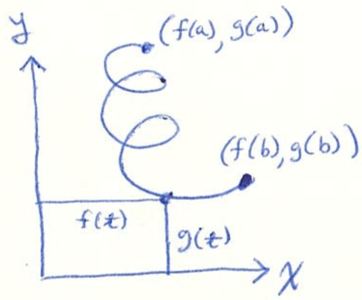
Chapter 12: Parametric and Polar Curves

Section 12.1: Parametric Equations



There is a story about Descartes as a young man. While confined to his bed by illness he saw a bug moving on the ceiling and realized that its ~~position at time t~~ motion could be described by two functions $x = f(t)$ and $y = g(t)$ giving its distance from the two walls at time t. This is the idea of parametric representation of a curve.

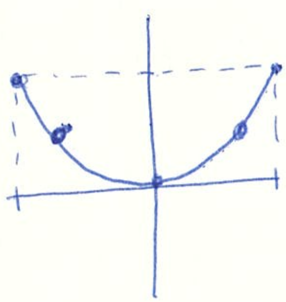


Parametric Representation of a Curve

A plane curve is represented by two functions

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \quad t \in [a, b]$$

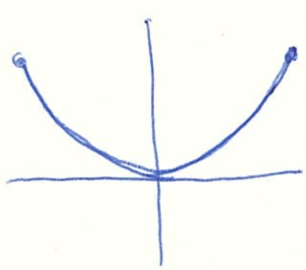
t is called the parameter



Example A

$$\left.\begin{aligned} x &= \sin t \\ y &= \sin^2 t \end{aligned}\right\} \quad t \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

t	x	y
$-\frac{\pi}{2}$	-1	1
$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
0	0	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	1



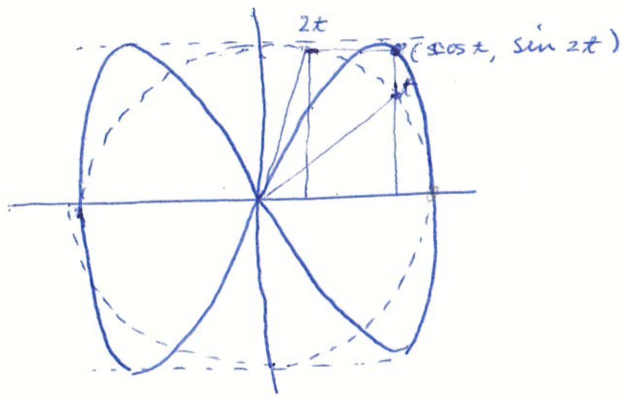
Example B

$$\left.\begin{aligned} x &= t \\ y &= t^2 \end{aligned}\right\} \quad t \in [-1, 1]$$

t	x	y
-1	-1	1
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$
0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
1	1	1

Remark: A and B give two parametric representations with the same graph.

Example C $\left.\begin{array}{l} x = \cos t \\ y = \sin 2t \end{array}\right\}$ $0 \le t \le 2\pi$



t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{\sqrt{3}}{2} \approx 0.85$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.7$	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2} = 0.85$
$\frac{\pi}{2}$	0	0

⋮

Eliminating the parameter

You will often want to get rid of the parameter, and express the curve as a single equation containing x and y. Sometimes this will simplify the situation; sometimes it will make it more complicated.

Example A $\left.\begin{array}{l} x = \sin t \\ y = \sin^2 t \end{array}\right\}$ $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\Rightarrow$ $y = x^2$, $-1 \le x \le 1$

Example C $\left.\begin{array}{l} x = \cos t \\ y = \sin 2t \end{array}\right\}$ $0 \le t \le 2\pi$

$$y = \sin 2t = 2 \sin t \cos t$$

$$y^2 = 4 \sin^2 t \cos^2 t = 4(1-\cos^2 t)\cos^2 t$$

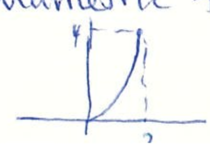
$$y^2 = 4(1-x^2)x^2$$

$$y = \pm 2x\sqrt{1-x^2}$$

Introducing A Parameter

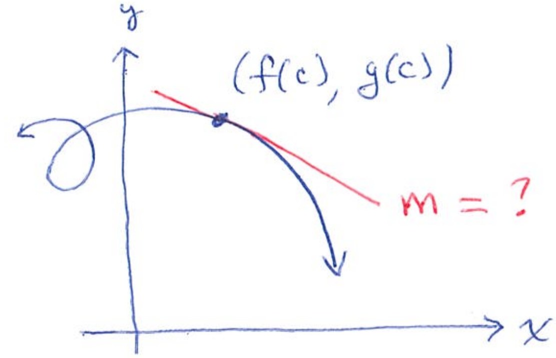
Ex B Write $y = x^2$ for $0 \le x \le 2$ in parametric form

Ans. $\left.\begin{array}{l} x = t \\ y = t^2 \end{array}\right\}$ $0 \le t \le 2$



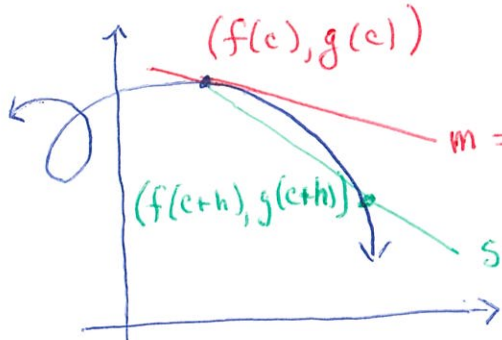
Slope

$$\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\}$$



Question:

What is the slope?



secant slope $= \dfrac{g(c+h) - g(c)}{f(c+h) - f(c)}$

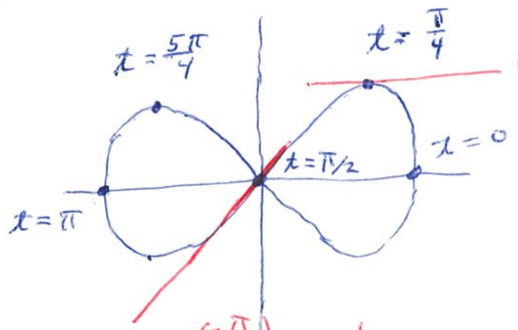
Ans $m = \lim\limits_{h \to 0} \dfrac{g(c+h) - g(c)}{f(c+h) - f(c)} = \lim\limits_{h \to 0} \dfrac{\frac{g(c+h) - g(c)}{h}}{\frac{f(c+h) - f(c)}{h}} = \boxed{\dfrac{g'(c)}{f'(c)}}$

Theorem Given a parametric curve $\begin{cases} \cancel{x} = f(x) \\ y = g(x) \end{cases}$

the slope at $(f(t), g(t))$ is $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{h'(t)}$

Example $\begin{cases} x = \cos(t) \\ y = \sin(2t) \end{cases}$

Slope at $(\cos(t), \sin(2t))$ is $m = \dfrac{2\cos(2t)}{-\sin(t)}$

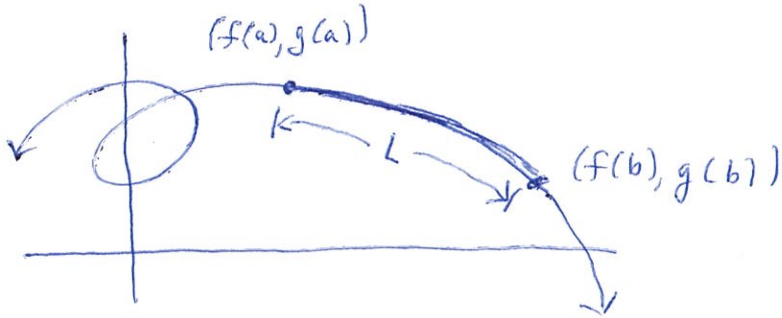


$m = \dfrac{2\cos(2\frac{\pi}{4})}{-\sin(\pi/4)} = 0$

$m = \dfrac{2\cos(2\frac{\pi}{2})}{-\sin(\frac{\pi}{2})} = \dfrac{-1}{-1} = 1$

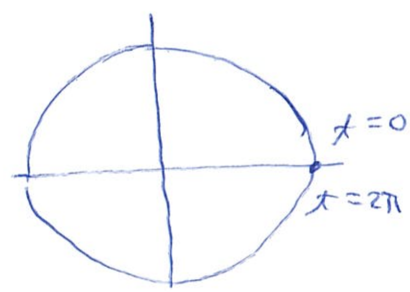
Arc Length

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$



Shown in text: $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$

Example $\left.\begin{aligned} x &= \cos(t) \\ y &= \sin(t) \end{aligned}\right\}$ $\underset{\uparrow \atop a}{0} \le t \le \underset{\uparrow \atop b}{2\pi}$



$$L = \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t)}\, dt = \int_0^{2\pi} dt$$

$$= \left[t\right]_0^{2\pi} = 2\pi - 0 = 2\pi \text{ units.}$$

circumference of unit circle!