§ 11.4 Working with Taylor Series

We finish the course with a few notes on Taylor series.

First, remember and internalize the basic series:

(1) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ on $(-\infty, \infty)$

(2) $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ on $(-\infty, \infty)$

(3) $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ on $(-\infty, \infty)$

(4) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ on $(-1, 1)$

(5) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ on $(-1, 1)$

integrate

(5) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ on $(-1, 1]$

(6) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots$ on $(-1, 1)$

integrate

(7) $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$ on $[-1, 1]$

Limits Sometimes series can help with limits.

Ex $\lim_{x \to 0} \frac{\tan^{-1}(x) - x}{x^3}$ ← form $\frac{0}{0}$, but L'Hopital's rule looks like a mess :(

$$= \lim_{x \to 0} \frac{1}{x^3}\left(\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) - x\right)$$

$$= \lim_{x \to 0} \frac{1}{x^3}\left(-\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)$$

$$= \lim_{x \to \infty}\left(-\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \frac{x^6}{9} + \dots\right)$$

$$= -\frac{1}{3} + 0 + 0 + 0 + 0 \dots \quad = \boxed{\frac{1}{3}}$$

Integrals Series can help with integrals.

Ex $\int \tan^{-1}(x^2)\,dx$ ← Hard to do! Earlier techniques don't apply

$$\tan^{-1}(u) = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \dots$$

$$\tan^{-1}(x^2) = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \frac{(x^2)^7}{7} + \dots = \frac{x^2}{1} - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots$$

$$\int \tan^{-1}(x)^2 dx = \int\left(x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \dots\right)dx$$

$$= \frac{x^3}{3 \cdot 1} - \frac{x^7}{7 \cdot 3} + \frac{x^{11}}{11 \cdot 5} - \frac{x^{15}}{15 \cdot 7} + \dots + C = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2(2k+1)+1}}{(4k+3)(2k+1)} + C$$

Recognizing Functions from Series

Sometimes its useful to look at a power series and decide what function it represents.

Example $\displaystyle\sum_{k=0}^{\infty} 2^k x^{2k+1} = 1x + 2x^3 + 4x^5 + 8x^7 + \cdots$

What familiar function is this?

$$\sum_{k=0}^{\infty} 2^k x^{2k+1} = 2^0 \cdot x + 2^1 x^3 + 2^2 x^5 + 2^3 x^7 + 2^4 x^9 + \cdots$$

$$= \sqrt{2}^0 x^1 + \sqrt{2}^2 x^3 + \sqrt{2}^4 x^5 + \sqrt{2}^6 x^7 + \sqrt{2}^8 x^9 + \cdots$$

$$= x\left(1 + \sqrt{2}^2 x^2 + \sqrt{2}^4 x^4 + \sqrt{2}^6 x^6 + \sqrt{2}^8 x^8 + \cdots\right)$$

$$= x\left(1 + 2x^2 + 4x^4 + 8x^6 + 16x^8 + \cdots\right)$$

$2x^2$ $2x^2$ $2x^2$ $2x^2$

$$= x \frac{1}{1 - 2x^2} = \frac{x}{1 - 2x^2}$$

Looking at functions as power series as we have done here will be a key idea in many of your more advanced math classes.