Chapter 11 Power Series

Basic Idea: Let $f(x)$ be a complex function, one that can't be expressed with a combination of the operations $+, -, \times, \div$, like $f(x) = \cos(x)$, e^x, $\ln(x)$, etc.

Goal $$f(x) = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$f(x) \approx \sum_{k=0}^{n} c_k x^k = \underbrace{c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n}_{\text{polynomial}}$$

§ 11.1 Approximating Functions With Polynomials.

Two ingredients are needed to cary out this plan.

① Factorials

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2\cdot 1 = 2 \\ 3! &= 3\cdot 2\cdot 1 = 6 \\ 4! &= 4\cdot 3\cdot 2\cdot 1 = 24 \\ &\vdots \end{aligned}$$

Definition $$n! = n\underbrace{(n-1)(n-2)(n-3)\cdots 1}_{(n-1)!}$$

Formulas $$\boxed{\begin{aligned} n! &= n(n-1)! \\ \frac{n!}{n} &= (n-1)! \end{aligned}}$$

$\longrightarrow 1 = 1! = 1\cdot(1-1)! = 1\cdot 0! = 0!$

② Higher Derivatives

$$\begin{aligned} f^{(0)}(x) &= f(x) \\ f^{(1)}(x) &= f'(x) \\ f^{(2)}(x) &= f''(x) \\ &\vdots \end{aligned}$$

Goal Attained

<u>Definition</u> Given a function $f(x)$, its <u>Maclaurin series</u> is

$$p(x) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \frac{f^{(0)}(0)}{0!}x^0 + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \cdots$$

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$$

Thus $p(x)$ is a polynomial of "infinite degree".

We will soon see $f(x) = p(x)$ {under certain conditions}

For now, notice the derivatives of $f(x)$ and $p(x)$ agree at $x=0$, i.e.

$$p(0) = f(0)$$
$$p'(0) = f'(0)$$
$$p''(0) = f''(0)$$
$$p'''(0) = f'''(0)$$
$$\vdots$$

$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots$	$p(0) = f(0)$
$p'(x) = f'(x) + f''(0)x + \frac{f'''(0)}{2}x^2 + \cdots$	$p'(0) = f'(0)$
$p''(x) = f''(0) + f'''(0)x + \cdots$	$p''(0) = f''(0)$
$p'''(0) = f'''(0) + \cdots$	$p'''(0) = f'''(0)$

Example Maclaurin Series for $f(x) = e^x$

$$p(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{e^0}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$p(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$p_1(x) = 1 + x$

$p_2(x) = 1 + x + \frac{x^2}{2}$

$p_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$\vdots$