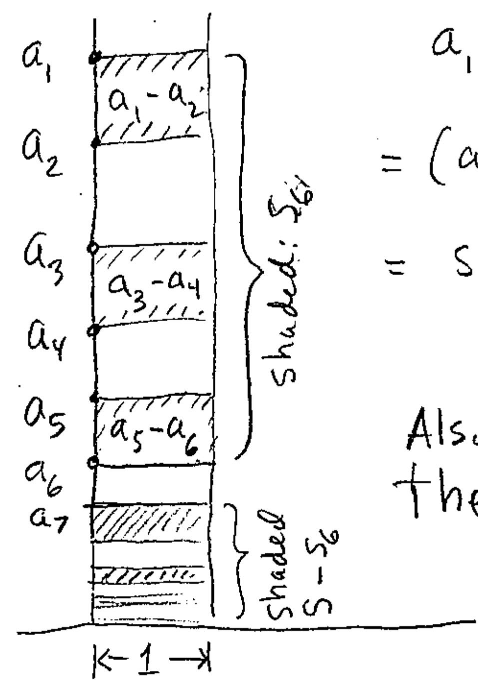
§ 10.6 Alternating Series

An alternating series is one that alternates signs

Examples
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$
- $\sum_{K=1}^{\infty} \left(-\frac{1}{2}\right)^n = -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots = \frac{-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = -\frac{1}{3}$

Theorem 10.16 (Alternating Series Test)
An alternating series $\sum_{K=1}^{\infty} (-1)^K a_K$ $\left(\text{or } \sum_{K=1}^{\infty} (-1)^{K+1} a_K\right)$ converges provided
1. $a_1 > a_2 > a_3 > \cdots$
2. $\lim_{K \to \infty} a_K = 0$

Reason: Suppose 1 and 2 hold



$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$$
$$= (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots$$
$$= \text{shaded area on right}$$

Also note: If $\sum (-1)^{K+1} a_i = S$ then $|S - S_n| \leq a_{n+1}$

Theorem 10.18 If $\sum_{K=1}^{\infty} (-1)^{K+1} a_K = S$, then $\underbrace{|S - S_n|}_{R_n} \leq a_{n+1}$

Recall Harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges

Theorem 10.17 The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots$ converges.

Proof Follows from alternating series test.

Example

Say $S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

Now, $S_{99} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{99} \approx 0.0698172$

How close is S_{99} to S? By Theorem 10.17,

$|S - S_{99}| < a_{100} = \frac{1}{100} = 0.01.$

Example $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}+1}$ ← converges by alternating series test!

Reason:

$$\frac{1}{\sqrt{1}+1} > \frac{1}{\sqrt{2}+1} > \frac{1}{\sqrt{3}+1} > \frac{1}{\sqrt{4}+1} > \cdots$$

and $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt{k}+1} = 0$

Absolute Convergence

What if a series has positive and negative terms, but it does not alternate.

Ex $\frac{1}{2!} - \frac{1}{3!} - \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} - \frac{1}{8!} + \cdots$

The notion of so-called absolute convergence helps.

Theorem 10.19

If $\sum |a_k|$ converges, then $\sum a_k$ converges.
If $\sum a_k$ diverges, then $\sum |a_k|$ diverges.

Ex $\sum a_k = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$
converges because $\sum |a_k|$ converges.

Definition

1. $\sum a_k$ converges absolutely if it converges and $\sum |a_k|$ converges
2. $\sum a_k$ converges conditionally if it converges but $\sum |a_k|$ diverges

Ex $1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \cdots$ ← converges
← converges absolutely

Ex $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$ ← converges
← converges conditionally.

Note any series with all positive terms that converges also converges absolutely.