§10.5 Comparison Tests

Today we introduce two new tests for convergence. Both of them require the sequence to have positive terms.

Theorem 10.14 Comparison Test

Suppose $\sum a_k$ and $\sum b_k$ have positive terms.

1. If $a_k \le b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges
2. If $b_k \le a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k+e^k}$ ← converge or diverge ?

Note $\dfrac{2}{k+e^k} \le \dfrac{2}{e^k}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{2}{e^k}$ is a convergent geometric series.

Thus $\displaystyle\sum_{k=1}^{\infty} \frac{2}{k+e^k}$ converges.

Ex $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5k-3}$ ← converge or diverge ?

$$\frac{1}{5k} \le \frac{1}{5k-3} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{5k} = 5\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges (harmonic series) thus $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5k-3}$ diverges.

Theorem 10.15 **Limit Comparison Test**

Suppose $\sum a_k$ and $\sum b_k$ have positive terms,

and $\lim\limits_{k \to \infty} \dfrac{a_k}{b_k} = L$

① If $0 < L < \infty$ then both series diverge or both converge

② If $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges

③ If $L = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges

Rough idea of why it works:

① Suppose $\lim\limits_{k \to \infty} \dfrac{a_k}{b_k} = L$ and $0 \leq L \leq \infty$.

Then for large k, $\dfrac{a_k}{b_k} \approx L \Rightarrow a_k \approx b_k L$

$$\Rightarrow \sum a_k \approx \sum b_k \cdot L = L \sum b_k$$

If one converges, so does other.
If one diverges, so does other.

② If $L = 0$ then $\sum a_k = 0$

③ If $L = \infty$ then $\sum a_k = \infty$.

Ex Does $\sum_{k=1}^{\infty} \frac{15}{k^2-k}$ converge or diverge?

Know $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent p-series

Note $\lim_{k\to\infty} \frac{\frac{15}{k^2-k}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{15k^2}{k^2-k} = 15$

$\boxed{\text{So both series converge.}}$ $\therefore$ $\boxed{\sum \frac{15}{k^2-k} \text{ converges}}$

Ex $\sum_{k=1}^{\infty} \frac{1}{2k-\sqrt{k}}$ converge or diverge?

Know $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

Note $\lim_{k\to\infty} \frac{\frac{1}{2k-\sqrt{k}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{2k-\sqrt{k}}$

$= \lim_{k\to\infty} \frac{1}{2-\frac{1}{2\sqrt{k}}} = \frac{1}{2-0} = \frac{1}{2}$

$\boxed{\text{Both series diverge.}}$ $\therefore$ $\boxed{\sum \frac{1}{2k-\sqrt{k}} \text{ diverges}}$