Section 10.4: Divergence and Integral Tests

Given an infinite series, two questions generally concern us

(1.) Does it converge or diverge?

(2) If it converges, what does it converge to?

This section presents several tools for answering the first question. Unfortunately, the second question can often be hard to answer. But if we determine that a series actually <u>diverges</u>, then the second question is moot. In this spirit we begin with a result that detects <u>divergence</u>.

<u>Theorem</u> If $\sum_{K=1}^{\infty} u_K$ converges, then $\lim_{K\to\infty} u_K = 0$.

The contrapositive form of this is more useful.

<u>Theorem</u> If $\lim_{K\to\infty} u_K \neq 0$ Then $\sum_{K=1}^{\infty} u_K$ diverges.

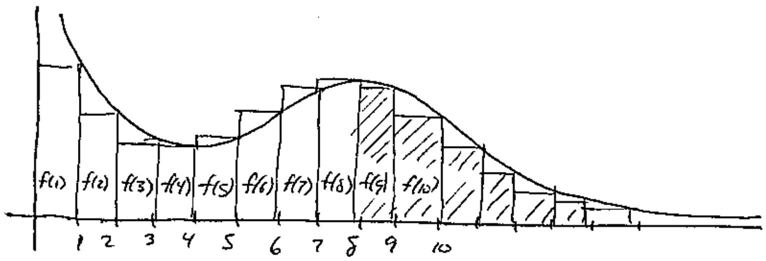
<u>Ex</u> $\sum_{K=1}^{\infty} k\sin(\frac{1}{k})$ $\leftarrow$ <u>diverges.</u> Reason:

$$\lim_{k\to\infty} k\sin\left(\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = \lim_{k\to\infty} \frac{\cos\left(\frac{1}{k}\right)\frac{-1}{k^2}}{-\frac{1}{k^2}} = 1 \neq 0$$

<u>Warning</u>: If $\lim_{k\to\infty} u_K = 0$, $\sum_{K=1}^{\infty} u_K$ may converge or diverge.

$$\left.\begin{array}{ll} \sum_{K=1}^{\infty} \frac{1}{K} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots & \text{(diverges)} \\ \sum_{K=1}^{\infty} \frac{1}{2^K} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1/2}{1-1/2} = 1 & \text{(converges)} \end{array}\right\} \lim_{K\to\infty} \frac{1}{K} = 0 = \lim_{K\to\infty} \frac{1}{2^K}.$$

Next, a test of convergence coming from this picture.



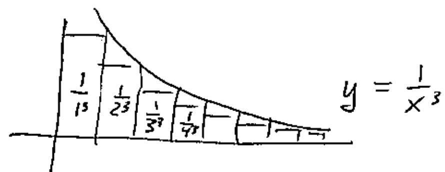
$$\sum_{k=8}^{\infty} f(k) \le \int_{8}^{\infty} f(x)\,dx$$

<u>The Integral Test</u> Suppose $\sum_{k=*}^{\infty} f(k)$ has positive terms and decreases on $[a, \infty)$

If $\int_a^{\infty} f(x)\,dx$ converges, then $\sum_{k=*}^{\infty} f(k)$ converges.

If $\int_a^{\infty} f(x)\,dx$ diverges, then $\sum_{k=*}^{\infty} f(k)$ diverges.

<u>Ex</u> $\sum_{k=1}^{\infty} \frac{1}{k^3}$ <u>converges</u> because



$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{l \to \infty} \int_1^{l} \frac{1}{x^3}\,dx = \lim_{l \to \infty} \left[\frac{-1}{2x^2}\right]_1^{l} = \lim_{l \to \infty} \left[\frac{-1}{2l^2} - \frac{-1}{2}\right] = \frac{1}{2} \text{ converges}$$

<u>Ex</u> $\sum_{k=1}^{\infty} \frac{1}{k}$ <u>diverges</u> because $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{l \to \infty} \int_1^{l} \frac{1}{x}\,dx = \lim_{l \to \infty} \ln l$ diverges.

<u>Definition</u> $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ is called a <u>p-series</u>.

<u>Theorem</u> $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $0 \le p \le 1$

<u>Reason</u> : We know $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges if $0 \le p \le 1$ and converges if p

<u>Ex</u> $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$ <u>converges</u>

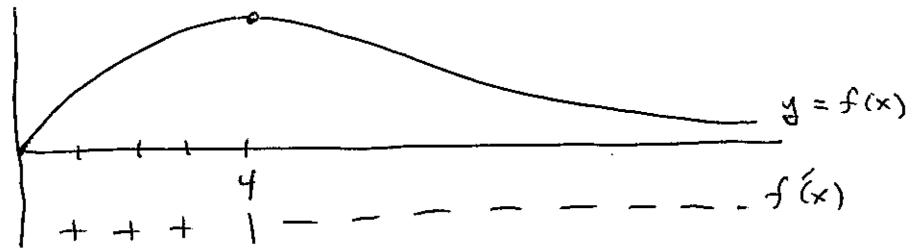
<u>Ex</u> $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots +$ <u>diverges</u>

Ex Does $\sum_{k=1}^{\infty} k e^{-\frac{k}{4}}$ converge or diverge?

$f(x) = x e^{-\frac{x}{4}}$

$f'(x) = e^{-\frac{x}{4}} + x e^{-\frac{x}{4}} \frac{1}{4}$

$= e^{-\frac{x}{4}}\left(1 + \frac{x}{4}\right)$



Note $f(x) = xe^{-\frac{x}{4}}$ is positive and decreases on $[4, \infty)$

$$\int_4^{\infty} x e^{-\frac{x}{4}}\,dx = \lim_{l \to \infty} \int_4^{l} x e^{-\frac{x}{4}}\,dx = \lim_{l \to \infty} \left[-e^{-\frac{x}{4}}(4x+16)\right]_4^{l}$$

$$= \lim_{l \to \infty}\left(-e^{-\frac{l}{4}}(4l+16) + e^{-\frac{4}{4}}(4 \cdot 4 + 16)\right)$$

$$= \lim_{l \to \infty}\left(\frac{-4l}{e^{l/4}} + \frac{16}{e^{l/4}} + \frac{32}{e}\right) = \lim_{l \to \infty} \frac{-4}{\frac{1}{4}e^{l/4}} + 0 + \frac{32}{e}$$

$$= \frac{32}{e} \quad \text{(converges)}$$

$$\int x e^{-x/4}\,dx = -4xe^{-\frac{x}{4}} - \int -4e^{-\frac{x}{4}}\,dx = -4xe^{-\frac{x}{4}} + 4\int e^{-\frac{x}{4}}\,dx$$

$$= -4xe^{-\frac{x}{4}} + 16e^{-\frac{x}{4}}$$

$u = x \quad dv = e^{-\frac{x}{4}}dx$

$du = dx \quad v = -4e^{-\frac{x}{4}}$

$$= -e^{-\frac{x}{4}}(4x + 16)$$

Now for some more useful results.

Theorem Suppose $\sum u_k$ and $\sum v_k$ are convergent. Then

(a) $\sum (u_k \pm v_k) = \sum u_k \pm \sum v_k$

(b) $\sum c u_k = c \sum u_k$

(c) Deleting a finite # of terms from a sequence does not affect convergence or divergence.

Ex $$\sum_{k=0}^{\infty}\left(\frac{5}{3^k} + \frac{(-1)^k}{2^k}\right) = \sum_{k=0}^{\infty} 5\left(\frac{1}{3}\right)^k + \sum_{k=0}^{\infty}\left(\frac{-1}{2}\right)^k = \frac{5}{1-\frac{1}{3}} + \frac{1}{1-\frac{-1}{2}}$$

$$= \frac{5}{\frac{2}{3}} + \frac{1}{\frac{3}{2}} = \frac{15}{2} + \frac{2}{3} = \frac{45}{6} + \frac{4}{6} = \frac{49}{6}$$

Ex $$\sum_{k=1}^{\infty}\left(\frac{1}{2^k} + \frac{3}{k^2}\right) = \underbrace{\sum_{k=1}^{\infty}\frac{1}{2^k}}_{\text{convergent geometric}} + 3\underbrace{\sum_{k=1}^{\infty}\frac{1}{k^2}}_{\text{convergent p-series}} \leftarrow \text{converges.}$$

Ex $$\frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \frac{1}{103} + \frac{1}{104} + \cdots \leftarrow \text{diverges.}$$

(harmonic series with 1st 99 terms deleted.)