

Section 10.3 Infinite Series

An infinite sequence is an ∞ list of #'s $\frac{1}{2} \frac{1}{4} \frac{1}{8} \frac{1}{16} \dots$

An infinite series is an ∞ sum of #'s

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = 1$$

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + u_4 + u_5 + \dots = S = ?$$

When does it make sense to say an ∞ series equals a #?
To answer this question we will make a definition.

Given an ∞ series $\sum_{k=1}^{\infty} u_k$ we define:

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$

$$S_4 = u_1 + u_2 + u_3 + u_4$$

\vdots

$$S_n = u_1 + u_2 + u_3 + \dots + u_n = \sum_{k=1}^n u_k \quad \leftarrow n^{\text{th}} \text{ partial sum}$$

$$\{S_1, S_2, S_3, S_4, \dots, S_n, \dots\} = \{S_k\}_{k=1}^{\infty} \quad \leftarrow \text{sequence of partial sums}$$

If $\{S_n\}_{n=1}^{\infty}$ has limit S then $\sum_{k=1}^{\infty} u_k = S$,

and we say the series converges to S .

Otherwise we say the series diverges.

Ex $\sum_{k=1}^{\infty} \frac{6}{10^k} = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \dots$

$$S_1 = .6$$

$$S_2 = .6 + .06 = .66$$

$$S_3 = .6 + .06 + .006 = .666$$

\vdots

$$S_n = .6 + .06 + \dots + .000\dots06 = .666\dots6$$

$$\{S_n\} = .6 \ .66 \ .666 \ .6666 \ .66666 \dots$$

$$\lim_{n \rightarrow \infty} S_n = .666666\bar{6} = \frac{2}{3}$$

$$\text{Therefore } \sum_{k=1}^{\infty} \frac{6}{10^k} = \frac{2}{3}$$

Important Example

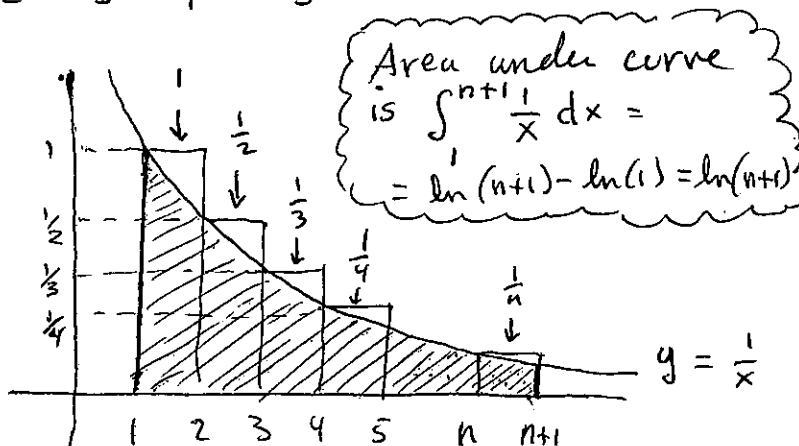
Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

Note:

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} > \ln(1+n)$$

$$\therefore \lim_{n \rightarrow \infty} S_n \geq \lim_{n \rightarrow \infty} \ln(1+n) = \infty$$

\therefore Harmonic Series diverges



Ex $\sum_{k=1}^{\infty} (-1)^k = -1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 + 1 \dots = ?$

$$\left. \begin{array}{l} S_1 = -1 \\ S_2 = 0 \\ S_3 = -1 \\ S_4 = 0 \\ S_5 = -1 \\ \vdots \end{array} \right\}$$

$\lim_{n \rightarrow \infty} S_n$ DNE, $\therefore \sum_{k=1}^{\infty} (-1)^k$ diverges

Theorem 10.8

- ① $\sum a_k = A \Rightarrow \sum c a_k = c \sum a_k = c A$
- ② $\sum a_k = A$ and $\sum b_k = B \Rightarrow \sum (a_k + b_k) = \sum a_k + \sum b_k = A + B$
- ③ If $\sum a_k$ diverges, and $c \neq 0$ then $\sum c a_k$ diverges
- ④ If one of $\sum a_k$ and $\sum b_k$ diverges, then $\sum (a_k \pm b_k)$ diverges
- ⑤ $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ either both converge or both diverge.

Telescoping Series

$$\text{Ex } \sum_{k=1}^{\infty} \left(\frac{1}{k+3} - \frac{1}{k+4} \right) = \lim_{n \rightarrow \infty} S_n$$

$$= \lim_{n \rightarrow \infty} \left(\left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \left(\frac{1}{6} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{8} \right) + \dots + \left(\frac{1}{n+3} - \frac{1}{n+4} \right) \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{4} - \frac{1}{n+4} \right) = \boxed{\frac{1}{4}}$$

$$\text{Ex } \sum_{k=2}^{\infty} \frac{2}{k^2-1} = \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} + \dots = ?$$

$$\frac{2}{k^2-1} = \frac{2}{(k+1)(k-1)} = \frac{A}{k+1} + \frac{B}{k-1}$$

$2 = A(k-1) + B(k+1)$

$\Rightarrow \begin{cases} B=1 \\ A=-1 \end{cases}$

$$\therefore \boxed{\frac{2}{k^2-1} = \frac{1}{k-1} - \frac{1}{k+1}}$$

$$\sum_{k=2}^{\infty} \frac{2}{k^2-1} = \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) = \lim_{n \rightarrow \infty} S_n$$

$$= \lim_{n \rightarrow \infty} \left(\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots + \left(\frac{1}{n-2} - \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} + \frac{1}{n+1} \right) = 1 + \frac{1}{2} + 0 + 0 =$$

$$= \boxed{\frac{3}{2}}$$