Chapter 7

Discrete Probability

An urban legend has it that one Friday a weatherman announced "*There's a 50% chance of rain on Saturday, and a 50% chance of rain on Sunday, so there's a 100% chance of rain this weekend.*" Obviously he was wrong, because under the circumstances there's still a chance of no rain at all over the weekend. But what is the correct answer?

Here is one approach to the answer. Make a set of four length-2 lists:

$$S = \big\{ \text{RR}, \quad \text{RN}, \quad \text{NR}, \quad \text{NN} \big\}.$$

This set encodes the four possible outcomes for the weather over the weekend. The first letter of each list is either R or N, depending on whether there is Rain or No-rain on Saturday. The second letter is either R or N depending on whether or not there is rain on Sunday. Thus RN means rain on Saturday and no rain on Sunday; NR means no rain on Saturday but rain on Sunday; RR means rain both days; and NN means no rain over the weekend.

The information suggests that each outcome RR, RN, NR and NN is equally likely to occur: There is a 25% chance of RR, a 25% chance of RN, a 25% chance of NR, and a 25% chance of NN.

We want to determine the chance of rain over the weekend. The event of rain over the weekend corresponds to the subset $\big\{\text{RR}, \text{RN}, \text{NR}\big\} \subseteq S$.

$$S = \big\{ \text{RR}, \quad \text{RN}, \quad \text{NR}, \quad \text{NN} \big\}$$

Thus rain over the weekend will occur in three out of four equally likely outcomes, so the weatherman should have said there is a $\frac{3}{4} = 75\%$ chance of rain over the weekend.

This chapter is about probability and computing the probabilities of events. The above example sets up the main ideas and definitions that are needed. Given a situation with a finite number of possible outcomes (like whether or not there's rain over the weekend), its *sample space* is the set S of all possible outcomes, and an *event* (like rain over the weekend) is a subset of S. Let's set up these ideas carefully.

7.1 Sample Spaces, Events and Probability

In the study of probability, an **experiment** is an activity that produces one of a number of different outcomes that cannot be known in advance. The **sample space** of the experiment is the set S of all possible outcomes. An **event** is a subset $E \subseteq S$. We say the **event occurs** if the experiment is performed and the outcome is an element of E.

One example of an experiment was described on the previous page: Observe whether it rains on each day of a weekend, and record the result as one of RR, RN, NR or NN. The sample space of this experiment is the set $S = \{\text{RR, RN, NR, NN}\}$. The event of rain over the weekend is the subset $E = \{\text{RR, RN, NR}\} \subseteq S$. If we perform the experiment and the outcome is one RR, RN or NR, then we say the event E occurs.

There are numerous other events associated with this experiment. The event of rain on Saturday is the subset $E' = \{\text{RR, RN}\} \subseteq S$. Here are some other events $E \subseteq S = \{\text{RR, RN, NR, NN}\}$ for this experiment.

Event		probability of event
Rain over the weekend:	$E = \{\text{RR, RN, NR}\}$	$p(E) = \frac{\lvert E\rvert}{\lvert S\rvert} = \frac{3}{4} = 75\%$
Rain on Sunday:	$E = \{\text{RR, NR}\}$	$p(E) = \frac{\lvert E\rvert}{\lvert S\rvert} = \frac{2}{4} = 50\%$
No rain over weekend:	$E = \{\text{NN}\}$	$p(E) = \frac{\lvert E\rvert}{\lvert S\rvert} = \frac{1}{4} = 25\%$
Rain on just one day:	$E = \{\text{RN, NR}\}$	$p(E) = \frac{\lvert E\rvert}{\lvert S\rvert} = \frac{1}{2} = 50\%$
Nothing happens:	$E = \emptyset$	$p(E) = \frac{\lvert E\rvert}{\lvert S\rvert} = \frac{0}{4} = 0\%$
Something happens:	$E = \{\text{RR, RN, NR NN}\}$	$p(E) = \frac{\lvert E\rvert}{\lvert S\rvert} = \frac{4}{4} = 100\%$

The **probability** or **chance** of an event is the likelihood of its occurring when the experiment is performed. The probability of an event is a number from 0 to 1 (that is, from a 0% chance of occurring to a 100% chance of occurring). We denote the probability of E as $p(E)$. Thus, in the experiment of recording the weather over the weekend when there is a 50% chance of rain on each day, the probability of the event $E = \{\text{RR, RN, NR}\}$ is $p(E) = 75\%$, as calculated on the previous page.

It is worth repeating and emphasizing that we will be concerned exclusively with experiments for which the sample space is finite, that is, *discrete*. The study of probability in such a setting is called **discrete probability**. (For an example of an experiment with a non-discrete sample space, consider measuring the rainfall on a given day. An outcome could be any of infinitely many values of x centimeters. We will not consider such experiments.)

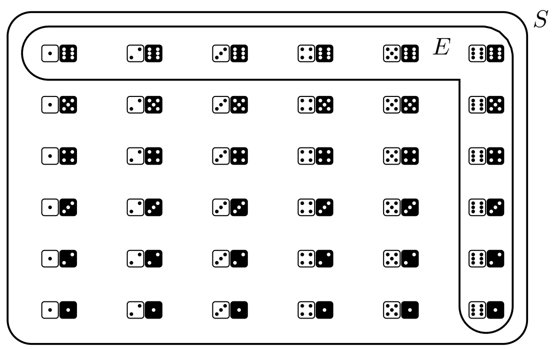
In many cases, all outcomes in a (finite) sample space are equally likely to occur. This is the case in the above weekend weather experiment, where each outcome RR, RN, NR, or NN has a 25% chance of occurring. In such a situation, an event E occurs in $|E|$ out of $|S|$ equally likely outcomes, so its probability is $p(E) = \frac{|E|}{|S|}$. See the right-hand column of the above table.

This type of reasoning leads to a formula for the probability of an event when all outcomes in a sample space are equally likely to occur.

> **Fact 7.1.** In an experiment where all outcomes in the sample space S are equally likely to occur, the probability of an event $E \subseteq S$ is
> $$p(E) = \frac{|E|}{|S|}.$$

Example 7.1. You have two dice, a white one and a black one. You roll both of them. What is the probability that at least one of them will be a six?

Solution. The sample space S is drawn below, showing the 36 equally likely outcomes. The event $E \subseteq S$ of at least one six is also shown.



Note that you will get at least one six in $|E| = 11$ out of $|S| = 36$ equally likely outcomes, so Fact 7.1 says the probability of getting at least one six is

$$p(E) = \frac{|E|}{|S|} = \frac{11}{36} = 0.30\overline{5} = 30.\overline{5}\%.$$

This means that if you roll the pair of dice, say, 100 times, you should expect to get at least one six on about 30 of the rolls. Try it. ✍

Fact 7.1 applies only to situations in which all outcomes in a sample space are equally likely to occur. For an example of an experiment that does not meet this criterion, imagine that one of the dice in Example 7.1 was weighted so that it was more likely to land on six. Then the outcome ⚅⚅ would be more likely than the outcome (say) ⚀⚀, and Fact 7.1 would not apply. In such a case $p(E)$ would be greater than $30.\overline{5}\%$. We will treat this kind of situation in Section 7.4. Until then, all of our experiments will have outcomes that are equally likely, and we will use Fact 7.1 freely.

Example 7.2. You toss a coin three times in a row. What is the probability of getting at least one tail?

Solution. Denote a typical outcome as a length-3 list such as HTH, which means you rolled a head first, then a tail, and then a head. Here is the sample space S and the event E of at least one tail:

$$S = \{ \text{HHH}, \overbrace{\text{HHT, HTH, HTT, THH, THT, TTH, TTT}}^{E} \}$$

The chance of getting at least one tail is $p(E) = \dfrac{|E|}{|S|} = \dfrac{7}{8} = 0.875 = \mathbf{87.5}\%$. ✍

Example 7.3. You deal a 5-card hand from a shuffled deck of 52 cards. What is the probability that all five cards are of the same suit?

Solution. The sample space S consists of all possible 5-card hands. Such a hand is a 5-element subset of the set of 52 cards, so we could begin writing out S as something like

$$S = \Big\{ \big\{ \boxed{\clubsuit^{7}}, \boxed{\clubsuit^{2}}, \boxed{\heartsuit^{3}}, \boxed{\spadesuit^{A}}, \boxed{\diamondsuit^{5}} \big\}, \quad \big\{ \boxed{\heartsuit^{8}}, \boxed{\heartsuit^{2}}, \boxed{\heartsuit^{K}}, \boxed{\heartsuit^{A}}, \boxed{\heartsuit^{5}} \big\}, \ \ldots\ldots \Big\}.$$

However, this is too big to write out conveniently in its entirety. But note that $|S|$ is the number of ways to select 5 cards from 52 cards, so

$$|S| = \binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960.$$

Now consider the event $E \subseteq S$ consisting of all 5-card hands in S that are of the same suit. We can compute $|E|$ using the addition principle (Fact 6.2 on page 116). The set E can be divided into four parts: the hands that are all hearts, the hands that are all diamonds, the hands that are all clubs and the hands that are all spades.

As the deck has 13 heart cards, the number of 5-card hands that are all hearts is $\binom{13}{5} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1287$. For the same reason, the number of 5-card hands that are all diamonds is also 1287. This is also the number of 5-card hands that are all clubs, and the number of 5-card hands that are all spades. By the addition principle, $|E| = 1287 + 1287 + 1287 + 1287 = 5148$.

Thus the probability that all cards in the hand are of the same suit is thus $p(E) = \dfrac{|E|}{|S|} = \dfrac{5148}{2{,}598{,}960} \approx 0.00198 = \mathbf{0.198}\%$.

So in playing cards, you should expect to be dealt a 5-card hand of the same suit only approximately 2 out of 1000 times. ✍

Exercises for Section 7.1

For each problem, write out the sample space S (or describe it if it's too big to write out) and find $|S|$. Then write out or describe the relevant event E. Find $p(E) = \frac{|E|}{|S|}$. You may need to use various counting techniques from Chapter 6

1. You have two dice, a white one and a black one. You toss both of them together. Find the probability that they both show the same number.

2. Toss a coin and then roll a dice. You win \$1 if the coin is heads or the dice is ⚃. What are your chances of winning?

3. You toss a coin and then roll a dice. You win \$1 if the the coin is heads or the dice is even. What are your chances of winning?

4. A card is randomly selected from a deck of 52 cards. What is the chance that the card is red but not a king?

5. A card is randomly selected from a deck of 52 cards. What is the chance that the card is red or a king?

6. A black dice and a white dice are tossed simultaneously. What are the chances that they both land on 6?

7. Toss a dice 5 times. What is the probability that you don't get any ⚅'s?

8. Toss a dice 6 times. What is the probability that exactly three tosses are even?

9. Toss a dice 5 times. What is the probability that you will get the same number on each roll? (i.e. ⚀⚀⚀⚀⚀ or ⚂⚂⚂⚂⚂, etc.)

10. Toss a dice 5 times. What is the probability that every roll is a different number?

11. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that they show different numbers?

12. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that the numbers add up to 7?

13. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that both show even numbers?

14. Toss a coin 8 times. What is the probability of getting exactly two heads?

15. Toss a coin 8 times. Find the probability that the first and last tosses are heads.

16. A hand of four cards is dealt off of a shuffled 52-card deck. What is the probability that all four cards are of the same color? (All red or all black.)

17. Five cards are dealt from a shuffled 52-card deck. What is the probability of getting three red cards and two clubs?

18. A coin is tossed 7 times. What is the probability that there are more tails than heads? What if it is tossed 8 times?

19. Alice and Bob each randomly pick an integer from 0 to 9. What is the probability they pick the same number? What is the probability they pick different numbers?

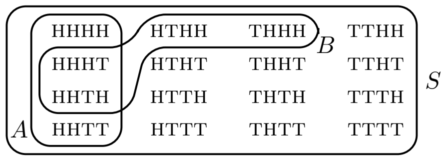
20. Alice and Bob each randomly pick an integer from 0 to 9. What is the probability that Alice picks an even number and Bob picks an odd number?

21. What is the probability that a 5-card hand dealt off a shuffled 52-card deck does not contain an ace?

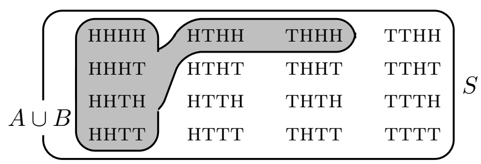
22. What is the probability that a 5-card hand dealt off a shuffled 52-card deck does not contain any red cards?

7.2 Combining Events

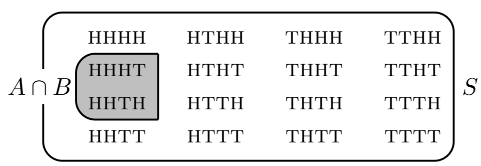
Now we begin combining events. To illustrate this, imagine tossing a coin four times in a row. Let A be the event "*The first two tosses are heads*," and let B be the event "*There are exactly three heads*." The sample space S is shown below, along with the events A and B. Note that $p(A) = \frac{|A|}{|S|} = \frac{4}{16} = 25\%$ and $p(B) = \frac{|B|}{|S|} = \frac{4}{16} = 25\%$.



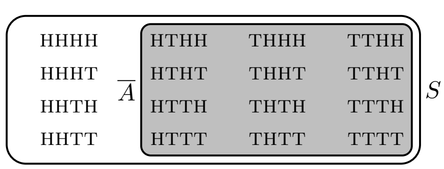
Now, the union $A \cup B$ is a subset of S, so it is an event. Think of it as the event "*The first two tosses are heads* **or** *there are exactly three heads*." This is diagramed below, and we see that $p(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{6}{16} = 37.5\%$.



Also, the intersection $A \cap B$ is a subset of S, so it is an event. It is the event "*The first two tosses are heads* **and** *there are exactly three heads*." This is diagramed below. Note that $p(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{2}{16} = 12.5\%$.



Finally, regard S as a universal set and consider the complement $\overline{A} \subseteq S$, drawn below. This is yet another event. It is the event "*It is not the case that the first two tosses are heads*." We have $p\left(\overline{A}\right) = \frac{|\overline{A}|}{|S|} = \frac{12}{16} = 75\%$.

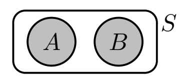


In general, if A and B are events in a sample space, then:

$A \cup B$ is the event	"A **or** B *occurs*,"
$A \cap B$ is the event	"A **and** B *occur*,"
$\overline{A}$ is the event	"A **does not** *occur*."

This section develops formulas for $p(A \cup B)$ and $p\left(\overline{A}\right)$, while the next section treats $p(A \cap B)$. These formulas will be useful because often a complex event has form $E = A \cup B$ or $E = \overline{A}$, where A and B (or $\overline{A}$) are easier to deal with than E. In such cases formulas for $p(A \cup B)$ and $p\left(\overline{A}\right)$ can be handy. But before stating them, we need to lay out a definition.

Definition 7.1. Two events A and B in a sample space S are **mutually exclusive** if $A \cap B = \emptyset$.



Mutually exclusive events have no outcomes in common: If one of them occurs, then the other does not occur. On any trial of the experiment, one of them may occur, or the other, or neither, but *never both*.

Events A and B from the previous page are *not* mutually exclusive, as $A \cap B = \{\text{HHHT, HHTH}\} \neq \emptyset$. You can toss a coin four times and have both events A: *First two tosses are heads*, and B: *Exactly three heads* occur.

Again, toss a coin four times. Say A is the event "*Exactly three tails*," and B is "*Exactly three heads*." These events are mutually exclusive. You could get three heads, or three tails, or neither (HHTT), but you cannot get three heads **and** three tails in the same four tosses.

Also, if E is any event in a sample space, then E and $\overline{E}$ are mutually exclusive, as $E \cap \overline{E} = \emptyset$. An event E cannot both happen and not happen.

Now we are ready to derive our formula for $p(A \cup B)$. We will get it using the formula $p(E) = |E|/|S|$ and the inclusion-exclusion principle (Fact 6.7, on page 134) that states $|A \cup B| = |A| + |B| - |A \cap B|$. Simply observe that

$$p(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \boxed{p(A) + p(B) - p(A \cap B)}.$$

Note that if A and B happen to be mutually exclusive, then $|A \cap B| = |\emptyset| = 0$, and we get simply $p(A \cup B) = p(A) + p(B)$.

For the formula for $p\left(\overline{A}\right)$, use $\overline{A} = S - A$ and note that

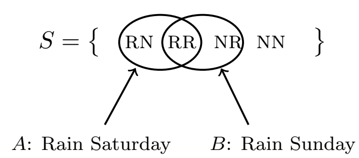
$$p\left(\overline{A}\right) = \frac{|\overline{A}|}{|S|} = \frac{|S - A|}{|S|} = \frac{|S| - |A|}{|S|} = \frac{|S|}{|S|} - \frac{|A|}{|S|} = \boxed{1 - p(A)}.$$

Rearranging $p\left(\overline{A}\right) = 1 - p(A)$ gives $p(A) = 1 - p\left(\overline{A}\right)$, also a useful formula. In summary, we have deduced the following facts.

Fact 7.2. Suppose A and B are events in a sample space S. Then:

1. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
2. $p(A \cup B) = p(A) + p(B)$ ....if A and B are mutually exclusive
3. $p(\overline{A}) = 1 - p(A)$
4. $p(A) = 1 - p(\overline{A})$

Recall our weatherman that we began the chapter with, the one who said that because there was a 50% chance of rain on Saturday and a 50% chance of rain on Sunday, then there was a 100% chance of rain over the weekend. He had only a hazy understanding of the events A: *Rain on Saturday*, and B: *Rain on Sunday*, and their union $A \cup B$: *Rain over the weekend*.



From the data $p(A) = 50\%$ and $p(B) = 50\%$ he concluded $p(A \cup B) = p(A) + p(B) = 50\% + 50\% = 100\%$. The problem is that A and B are not mutually exclusive, as $A \cap B = \{\text{RR}\} \neq \emptyset$. In essence he was using Formula 2 of Fact 7.2, above, when he should have used Formula 1. The correct chance of rain over the weekend, as given by Formula 1, is

$$\begin{aligned} p(A \cup B) &= p(A) + p(B) - p(A \cap B) \\ &= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|\{\text{RR}\}|}{|S|} \\ &= \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4} = 75\%. \end{aligned}$$

Of course we can also get this answer without the aid of the formula $p(A \cup B) = p(A) + p(B) - p(A \cap B)$. Just let $E = A \cup B$ be the event of rain over the weekend. Fact 7.1, which states $p(E) = \frac{|E|}{|S|}$, says $p(E) = \frac{3}{4} = 75\%$. But a word of caution is in order. Recall that Fact 7.1 is only valid in situations in which all outcomes in S are equally likely to occur. Such is the case in this rain-over-the-weekend example (and all other examples in the next three sections), so we do not get into trouble. But the point is that the formulas from Fact 7.2 turn out to hold even if not all outcomes in S are equally likely (even though we derived them under that assumption on the previous page). We will investigate this thoroughly in Section 7.4.

For now, let's do some examples involving Fact 7.2.

Example 7.4. Two cards are dealt from a shuffled deck of 52 cards. What is the probability that both cards are red or both cards are clubs.

Solution. Regard a 2-card hand as a 2-element subset of the set of 52 cards. So the sample space is the set S of 2-element subsets of the 52 cards.

$$S = \left\{ \left\{ \boxed{\clubsuit^{7}}, \boxed{\clubsuit^{2}} \right\},\ \left\{ \boxed{\heartsuit^{8}}, \boxed{\heartsuit^{2}} \right\},\ \left\{ \boxed{\spadesuit^{K}}, \boxed{\heartsuit^{2}} \right\},\ \left\{ \boxed{\clubsuit^{K}}, \boxed{\heartsuit^{2}} \right\}, \ldots \right\}.$$

Though this is too large to write out, we can compute $|S| = \binom{52}{2} = \frac{52 \cdot 51}{2} = 1326$.

We are asked to compute $p(E)$ where E is the event

E: *Both cards are red* **or** *both cards are clubs.*

We can decompose E as $E = A \cup B$ where A and B are the events

A: *Both cards are red*
B: *Both cards are clubs*

Thus $A = \left\{ \left\{ \boxed{\heartsuit^{K}}, \boxed{\diamondsuit^{K}} \right\}, \left\{ \boxed{\heartsuit^{K}}, \boxed{\heartsuit^{2}} \right\}, \left\{ \boxed{\diamondsuit^{8}}, \boxed{\heartsuit^{2}} \right\}, \left\{ \boxed{\diamondsuit^{Q}}, \boxed{\diamondsuit^{K}} \right\}, \ldots \right\} \subseteq S,$

and $B = \left\{ \left\{ \boxed{\clubsuit^{K}}, \boxed{\clubsuit^{5}} \right\}, \left\{ \boxed{\clubsuit^{7}}, \boxed{\clubsuit^{8}} \right\}, \left\{ \boxed{\clubsuit^{J}}, \boxed{\clubsuit^{Q}} \right\}, \left\{ \boxed{\clubsuit^{J}}, \boxed{\clubsuit^{7}} \right\}, \ldots \right\} \subseteq S.$

Note that these two events are mutually exclusive, as club cards are black. Further, $|A| = \binom{26}{2} = 325$ because to make a 2-card hand of red cards we have to choose 2 of the 26 red cards. Also, $|B| = \binom{13}{2} = 78$ because to make a 2-card hand of club cards we have to choose 2 of the 13 clubs. Using Formula 2 from Fact 7.2 as well as Fact 7.1 (page 165), our answer is

$$\begin{aligned} p(E) = p(A \cup B) &= p(A) + p(B) \\ &= \frac{|A|}{|S|} + \frac{|B|}{|S|} \\ &= \frac{325}{1326} + \frac{78}{1326} = \frac{403}{1326} \approx 0.3039 = \mathbf{30.39}\%. \end{aligned}$$

✍

You may prefer to solve this example without using Fact 7.2. Instead you can use Fact 7.1 combined with the addition principle, $|A \cup B| = |A| + |B|$, which holds when $A \cap B = \emptyset$ (as is the case here because A and B are mutually exclusive). Then compute the answer as

$$p(E) = p(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B|}{|S|} = \frac{325 + 78}{1326} = \frac{403}{1326} \approx \mathbf{30.39}\%.$$

But as noted on the previous page, there will be situations where Fact 7.2 is unavoidable. So it is not advisable to always bypass it. For now your best strategy is to become accustomed to it, but at the same time be on the lookout for alternate methods.

Example 7.5. Two cards are dealt from a shuffled deck of 52 cards. What is the probability both cards are red or both cards are face cards (J, K, Q)?

Solution. As before, the sample space is the set S of 2-element subsets of the 52 cards, and $|S| = \binom{52}{2} = 1326$:

$$S = \left\{ \left\{ \boxed{\clubsuit^{7}}, \boxed{\clubsuit^{2}} \right\}, \left\{ \boxed{\heartsuit^{8}}, \boxed{\heartsuit^{2}} \right\}, \left\{ \boxed{\spadesuit^{K}}, \boxed{\heartsuit^{2}} \right\}, \left\{ \boxed{\clubsuit^{K}}, \boxed{\heartsuit^{2}} \right\}, \ldots \right\}.$$

We are asked to compute $p(E)$ where E is the event

E: *Both cards are red or both cards are clubs.*

We can decompose E as $E = A \cup B$ where A and B are the events

A: *Both cards are red*
B: *Both cards are face cards*

Let's take a moment to diagram these two events, and their intersection.

$$A = \left\{ \left\{ \boxed{\heartsuit^{K}}, \boxed{\diamondsuit^{K}} \right\}, \left\{ \boxed{\heartsuit^{K}}, \boxed{\heartsuit^{2}} \right\}, \left\{ \boxed{\diamondsuit^{8}}, \boxed{\heartsuit^{2}} \right\}, \left\{ \boxed{\diamondsuit^{Q}}, \boxed{\diamondsuit^{K}} \right\}, \ldots \right\} \subseteq S.$$

$$B = \left\{ \left\{ \boxed{\heartsuit^{K}}, \boxed{\diamondsuit^{K}} \right\}, \left\{ \boxed{\spadesuit^{K}}, \boxed{\spadesuit^{J}} \right\}, \left\{ \boxed{\diamondsuit^{J}}, \boxed{\heartsuit^{Q}} \right\}, \left\{ \boxed{\diamondsuit^{Q}}, \boxed{\clubsuit^{J}} \right\}, \ldots \right\} \subseteq S.$$

$$A \cap B = \left\{ \left\{ \boxed{\heartsuit^{K}}, \boxed{\diamondsuit^{K}} \right\}, \left\{ \boxed{\diamondsuit^{K}}, \boxed{\diamondsuit^{Q}} \right\}, \left\{ \boxed{\diamondsuit^{J}}, \boxed{\heartsuit^{Q}} \right\}, \left\{ \boxed{\diamondsuit^{Q}}, \boxed{\diamondsuit^{J}} \right\}, \ldots \right\} \subseteq S.$$

Note that A and B are not mutually exclusive, because $A \cap B \neq \emptyset$. (It is possible for the two cards to be *both* red *and* both face cards.) Also,

$$|A| = \binom{26}{2} = \frac{26 \cdot 25}{2} = 325 \qquad \text{(choose 2 out of 26 red cards)}$$
$$|B| = \binom{12}{2} = \frac{12 \cdot 11}{2} = 66 \qquad \text{(choose 2 out of 12 face cards)}$$
$$|A \cap B| = \binom{6}{2} = \frac{6 \cdot 5}{2} = 15 \qquad \text{(choose 2 out of 6 red face cards)}$$

Using Fact 7.2, we get

$$\begin{aligned} p(E) &= p(A \cup B) = p(A) + p(B) - p(A \cap B) \\ &= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} \\ &= \frac{325}{1326} + \frac{66}{1326} - \frac{15}{1326} = \frac{376}{1326} \approx 0.2835 = \mathbf{28.35}\%. \end{aligned}$$

Answer: If two cards are dealt from a shuffled deck, the probability that both are red or both are face cards is **28.35%**. ✍

Example 7.6. Two cards from dealt off a shuffled deck of 52 cards. What is the probability they are not both red?

Solution. The sample space is the set S of 2-element subsets of the 52 cards, and $|S| = \binom{52}{2} = 1326$:

$$S = \Big\{ \underbrace{\{7\clubsuit, 2\clubsuit\}, \{K\spadesuit, 2\heartsuit\}, \{8\heartsuit, 2\spadesuit\} \cdots}_{E:\ \textit{Not both red}} \ \underbrace{\{K\heartsuit, 2\heartsuit\}, \{K\diamondsuit, 2\heartsuit\} \cdots}_{\overline{E}:\ \textit{Both red}} \Big\}.$$

We need to compute the probability of the event E: *Not both cards are red.* This event contains pairs of cards that are both black, as well as those for which one card is red and the other is black. The event $\overline{E}$ is simpler. It is the set of all elements of S for which it is not the case that not both cards are red. In other words, $\overline{E}$ is the event $\overline{E}$: *Both cards are red.*

It is easy to compute the cardinality of $\overline{E}$. It is $|\overline{E}| = \binom{26}{2} = 325$, the number of ways to choose 2 cards from the 26 red cards. Fact 7.2 now gives our solution:

$$\begin{aligned} p(E) &= 1 - p\left(\overline{E}\right) \\ &= 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{325}{1326} = \frac{1326 - 325}{1326} = \frac{1001}{1326} \approx 0.7549 = \mathbf{75.49}\%. \end{aligned}$$

That is the answer, but before moving on, let's redo the problem using a different approach. The event E is the union of the mutually exclusive events A: *Both cards are black*, and B: *One card is black and the other is red.* Here $|A| = \binom{26}{2} = 325$, the number of ways to choose 2 cards from the 26 blacks, while the multiplication principle says $|B| = 26 \cdot 26 = 676$ (chose a black card and then choose a red one). Fact 7.2 gives

$$\begin{aligned} p(E) &= p(A \cup B) = p(A) + p(B) \\ &= \frac{|A|}{|S|} + \frac{|B|}{|S|} = \frac{325}{1326} + \frac{676}{1325} = \frac{1001}{1326} = \mathbf{75.49}\%. \end{aligned}$$

Shuffle a 52-card deck; deal two cards, put them back. Repeat 100 times. On about 75 of the trials, not both cards will be red. ✍

But what if you shuffled the deck, dealt two cards, *but did not put them back.* Then you deal two cards from the remaining 50 cards. Is there still a 75.49% chance of not getting two reds? Does the outcome of the first trial affect the probability of the second? The next section investigates this kind of question.

Exercises for Section 7.2

1. A card is taken off the top of a shuffled 52-card deck. What is the probability that it is black or an ace?

2. What is the probability that a 5-card hand dealt off a shuffled 52-card deck contains at least one ace?

3. What is the probability that a 5-card hand dealt off a shuffled 52-card deck contains at least one red card?

4. A hand of five cards is dealt off a shuffled 52-card deck. What is the probability that the five cards are not all of the same suit?

5. You toss a fair coin 8 times. What is the probability that you do not get 4 heads?

6. A 4-card hand is dealt off a shuffled 52-card deck. What is the probability that the cards are all of the same color (i.e. all red or all black)?

7. Two cards are dealt off a shuffled 52-card deck.What is the probability that the cards are both red or both aces?

8. A coin is tossed six times. What is the probability that the first two tosses are heads or the last toss is a head?

9. A dice is tossed six times. You win \$1 if the first toss is a five or the last toss is even. What are your chances of winning?

10. A box contains 3 red balls, 3 blue balls, and 3 green ball. You reach in and grab 2 balls. What is the probability that they have the same color?

11. A dice is rolled 5 times. Find the probability that not all of the tosses are even.

12. Two cards are dealt off a well-shuffled deck. You win \$1 if either both cards are red or both cards are black. Find the probability of your winning.

13. Two cards are dealt off a well-shuffled deck. You win \$1 if the two cards are of different suits. Find the probability of your winning?

14. A dice is tossed six times. You win \$1 if there is at least one ⚄. Find the probability of winning.

15. A coin is tossed 5 times. What is the probability that the first toss is a head or exactly 2 out of the five tosses are heads?

16. In a shuffled 52-card deck, what is the probability that the top card is black or the bottom card is a heart?

17. In a shuffled 52-card deck, what is the probability that neither the top nor bottom card is a heart?

18. A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 15 marbles. What is the probability of getting 5 of each color?

19. A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 15 marbles. What is the probability that they are all the same color?

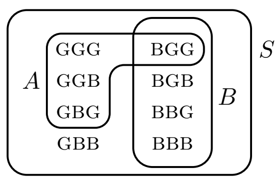
20. A 7-card hand is dealt off a shuffled standard 52-card deck. What is the probability that the hand consists entirely of red cards, or has no hearts?

7.3 Conditional Probability and Independent Events

Sometimes the probability of one event A will change if we know that another event B has occurred. This is what is known as *conditional probability.* Here is an illustration.

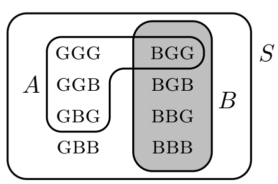
Imagine starting a family of three children. Assume the probability of having a boy is 50% and the probability of having a girl is 50%. What is more likely: the family has more girls than boys, or the oldest is a boy?

To decide, we write out the sample space for the family, listing the possible outcomes of having three children. Let GBB mean a girl was born first, then a boy, and then another boy. Likewise, BGB means a boy was born first, then a girl, then a boy, etc. The 8 equally-likely outcomes are shown below, with events A: *More girls than boys*, and B: *Oldest is a boy.*



Note $p(A) = \frac{|A|}{|S|} = \frac{4}{8} = 50\%$, and $p(B) = \frac{|B|}{|S|} = \frac{4}{8} = 50\%$, so more girls than boys is just as likely as the oldest being a boy.

But now imagine that the event B has occurred, so the oldest is a boy. *Now* what is the probability of more girls than boys? That is, what is $p(A)$? There are just four outcomes in event B, and for only one of them are there more girls than boys. Thus, given this new information (oldest is a boy) $p(A)$ has changed value to $p(A) = \frac{1}{4} = 25\%$.



So we have a situation in which $p(A) = 50\%$, but under the condition that B has occurred, then $p(A) = 25\%$. We express this as $p(A|B) = 25\%$, which we read as "*the conditional probability of A given that B has occurred is* 25%," or just *"the conditional probability of A given B is* 25%."

Definition 7.2. If A and B are two events in a sample space, then the **conditional probability of A given B**, written $P(A|B)$, is the probability that A will occur if B has already occurred.

Example 7.7. Toss a coin once. The sample space is $S = \{\text{H}, \text{T}\}$. Consider the events $A = \{\text{H}\}$ of getting a head and $B = \{\text{T}\}$ of getting a tail. Then $p(A) = p(B) = 50\%$, but $p(A|B) = 0$ because if B (tails) has happened, then A (heads) will not happen. Also $p(B|A) = 0$. Note that $p(A|A) = 1 = 100\%$. ✍

Example 7.8. Take one card from a shuffled deck, and then take another. Now you have two cards. Consider the following events.

A : *The first card is a heart* C : *The second card is a heart*
B : *The first card is black* D : *The second card is red*

Find $p(A)$, $p(B)$, $p(D)$, $p(C|A)$, $p(A|C)$, $p(A|B)$, $p(D|C)$ and $p(C|D)$.

Solution. All answers can be found without considering the sample space S. For example, $p(A) = \frac{13}{52} = \frac{1}{4}$ because in taking the first card there are 13 hearts among the 52 equally-likely cards. But to be clear, note that S is the set of all non-repetitive length-2 lists whose entries are cards in the deck. The first list entry is the first card drawn; the second entry is the second card. Taking 7♣ and then 2♣ is a different outcome than 2♣ and then 7♣.

$$S = \left\{ \boxed{7♣}\boxed{2♣},\ \boxed{2♣}\boxed{7♣},\ \boxed{K♠}\boxed{8♡},\ \boxed{5♣}\boxed{2♡},\ \boxed{5♣}\boxed{2♡},\ \ldots \right\}$$

Compare this to Example 7.6, where the outcomes were 2-element *sets*, not lists. In the present case $|S| = P(52, 2) = 52 \cdot 51 = 2652$.

As noted above, $p(A) = \frac{13}{52} = \frac{1}{4}$ because in dealing the first card there are 13 hearts in the 52 cards. Alternatively, the event $A \subseteq S$ consists of all the 2-elements lists whose first entry is a heart. There are 13 choices for the first entry, and then 51 of the remaining cards can be selected for the second entry. Thus $|A| = 13 \cdot 51$, and $p(A) = \frac{|A|}{|S|} = \frac{13 \cdot 51}{52 \cdot 51} = \frac{1}{4}$.

Similarly, $p(B) = \frac{26}{52} = \frac{1}{2}$ because in drawing the first card, there are 26 black cards out of 52. This could also be done by computing $|B|$, as above.

Evidently, $p(D) = \frac{1}{2}$, as half the cards are red. Alternatively, D consists of all the lists in S whose second entry is red, so by the multiplication principle, $|D| = 51 \cdot 26$. (Fill in the red second entry first, and then put one of the remaining 51 cards in the first entry.) Then $p(D) = \frac{|D|}{|S|} = \frac{51 \cdot 26}{52 \cdot 51} = \frac{1}{2}$.

Note $p(C|A) = \frac{12}{51}$ because if A has occurred, then a heart was drawn first, and there are 12 remaining hearts out of 51 cards for the second draw.

For $p(A|C)$, imagine the two cards have been dealt, one after the other, face down. The second card is turned over, and it is a heart. Event C has occurred. Now what is the chance that A occurred? That is, what is the chance that first card—when turned over—is a heart? It is not the second card, and there are 51 other cards, and 12 of them are hearts. Thus the chance that the first card is a heart is $\frac{12}{51}$, so $p(A|C) = \frac{12}{51}$.

Also, $p(A|B) = 0$ because if B occurs (first card black), then A (first card heart) is impossible. Finally, $p(D|C) = 100\%$, but $p(C|D) = 50\%$ because if the second card is red, there is a one in two chance that it is a heart. ✍

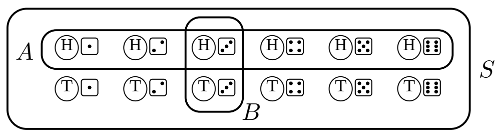
We will soon derive formulas for conditional probability, but they involve a definition that is motivated by the next example.

Example 7.9. Toss a coin and roll a dice. Consider the following events.

$$A : \textit{Coin is heads} \quad B : \textit{Dice is} ⚂$$

Find $p(A)$, $p(B)$, $p(A|B)$ and $p(B|A)$.

Solution. Common sense says getting ⚂ has no bearing the coin's outcome, and vice versa, so $P(A|B) = \frac{1}{2}$ and $p(B|A) = \frac{1}{6}$. Nonetheless, let's work it out carefully. The sample space S and events A and B are diagramed below.



We see that $P(A) = \frac{|A|}{|S|} = \frac{6}{12} = \frac{1}{2}$ and $P(B) = \frac{|B|}{|S|} = \frac{2}{12} = \frac{1}{6}$.

To find $p(A|B)$, imagine that B has occurred. Now what is the chance that A occurs? Only one of the two outcomes in B is heads, so $p(A|B) = \frac{1}{2}$.

To find $p(B|A)$, imagine that A has occurred. Now what is the chance that B occurs? Only 1 of the 6 outcomes in A has ⚂, so $p(B|A) = \frac{1}{6}$. ✍

In the above example, whether or not B happens has no bearing on the probability of A, and vice versa. We say that events A and B are *independent*.

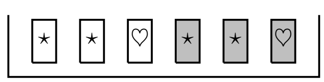
Definition 7.3. Two events A and B are **independent** if one happening does not change the probability of the other happening, that is, if $p(A) = p(A|B)$ and $p(B) = p(B|A)$. Otherwise they are **dependent**.

Thus events A and B in Example 7.9 are independent.

In Example 7.8 we dealt two cards off a deck. For events A: *First card* ♡, and B: *First card black*, we saw $p(A) = \frac{1}{4}$ and $p(A|B) = 0$. Because $\frac{1}{4} \neq 0$, A and B are dependent. (In fact, they also happen to be mutually exclusive.)

We began this section showing that in a family of three children, the event A : *More girls than boys* has $p(A) = 50\%$. But if B: *Oldest is a boy* occurs, then $p(A|B) = 25\%$. Here A and B are dependent. (But not mutually exclusive).

Example 7.10. A box contains six tickets, three white and three gray, and marked as shown below. You reach in and grab a ticket at random.



Consider events A: *Ticket is gray*, and B: *Ticket has a star on it.* Are these events independent or dependent?

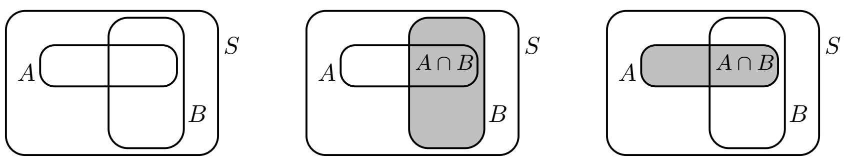
Solution. The chance of getting a gray ticket is $p(A) = \frac{3}{6} = \frac{1}{2}$. The chance of getting a star is $p(B) = \frac{4}{6} = \frac{2}{3}$.

If B occurs, then one of the four tickets with a star has been drawn. Half of these are gray, so $p(A|B) = \frac{1}{2}$, and this equals $p(A)$.

If A occurs, then one of the three gray tickets has been drawn. Two of these have stars, so $p(B|A) = \frac{2}{3}$, and this equals $p(B)$.

Thus A and B are independent. One of them happening does not change the probability of the other happening. ✍

Now we are going to derive general formulas for $p(A|B)$ and $p(B|A)$. Let A and B be two events in a sample space S, as shown below on the left.



If B occurs (shown shaded in the middle drawing) then any outcome in the shaded region could occur, so the shaded set B is like a new sample space. Now if also A occurs, this means some outcome in $A \cap B$ occurs. Note that $A \cap B \subseteq B$ is an event in B, so Fact 7.1 gives

$$p(A|B) = \frac{|A \cap B|}{|B|} = \frac{|A \cap B|}{|B|} \cdot \frac{\frac{1}{|S|}}{\frac{1}{|S|}} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{p(A \cap B)}{p(B)}.$$

Thus $p(A|B) = \frac{p(A \cap B)}{p(B)}$. Reversing the roles of A and B (and referring to the drawing on the above right) we also get $p(B|A) = \frac{p(A \cap B)}{p(A)}$. Cross-multiplying gives $p(A \cap B) = p(A|B) \cdot p(B)$ and $p(A \cap B) = p(A) \cdot p(B|A)$.

Thus we have formulas for not only $p(A|B)$ and $p(B|A)$, but also one for $p(A \cap B)$. Moreover, if A and B happen to be independent, then $p(A|B) = p(A)$, so the equation $p(A \cap B) = p(A|B) \cdot p(B)$ simplifies to $p(A \cap B) = p(A) \cdot p(B)$.

Fact 7.3. Suppose A and B are events in a sample space. Then:

1. $p(A|B) = \dfrac{p(A \cap B)}{p(B)}$

2. $p(B|A) = \dfrac{p(A \cap B)}{p(A)}$

3. $p(A \cap B) = p(A|B) \cdot p(B) = p(A) \cdot p(B|A)$

4. $p(A \cap B) = p(A) \cdot p(B)$ .... if A and B are independent.

In the earlier examples in this section, we found conditional probabilities $p(A|B)$ and $p(B|A)$ without the aid of the above formulas. In fact, it turns out that the above formulas 1 and 2 are of relatively limited use. But their consequences, formulas 3 and 4 are very useful, as they provide a method of computing $p(A \cap B)$, the probability that A and B both occur.

Example 7.11. Two cards are dealt off a deck. You win \$1 if the first card is red and the second card is black. What are your chances of winning?

Solution. Let A be the event "*The first card is red*," and let B be the event "*The second card is black*." We seek $p(A \text{ and } B)$, which is $p(A \cap B)$. Formula 3 above gives $p(A \cap B) = p(A) \cdot p(B|A) = \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{51} \approx 0.2549 = 25.49\%$. ✍

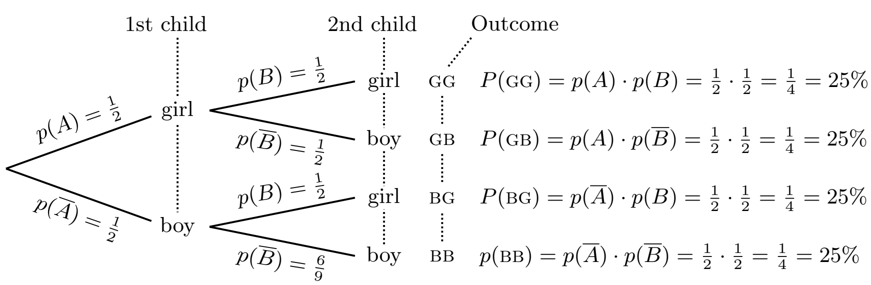
Example 7.12. A dice is rolled twice. You win \$1 if neither roll is ⚀. What are your chances of winning?

Solution. Let A be the event "*The first roll is not* ⚀," so $p(A) = \frac{5}{6}$. Let B be the event "*The second roll is not* ⚀," so $p(B) = \frac{5}{6}$. We seek $p(A \text{ and } B)$.

Events A and B are independent, because the result of the first roll does not influence the second. Formula 4 above gives $p(A \text{ and } B) = p(A \cap B) = p(A) \cdot p(B) = \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{36} = 69.\overline{4}\%$. ✍

Questions about conditional probability can sometimes be answered by a so-called **probability tree**. To illustrate this, suppose (as we assume in this chapter) that there is a 50-50 chance of a child being born a boy or a girl. Suppose a woman has two children. The events A: *First child is a girl,* and B: *Second child is a girl* are independent; whether or not the first child is a girl does not change the probability that the second child is a girl. Thus the chance that both children are girls is $p(A \cap B) = p(A) \cdot p(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 25\%$.

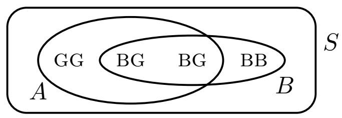
Notice that in this example the complements of A and B are the events $\overline{A}$: *First child is a boy,* and $\overline{B}$: *Second child is a boy.* The probability of the outcome GB (first child is a girl and the second is a boy) is thus $p(\text{GG}) = p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 25\%$. Similarly, we can find the probabilities of all four outcomes in the branches of the following tree.



This confirms our intuitive supposition that each outcome in the sample space $S = \{\text{GG, GB, BG, BB}\}$ has a 25% chance of occurring.

Example 7.13. You meet a woman who tells you that she has one sibling. What are the chances that the sibling is a brother?

Solution. Most of us would jump to the conclusion that the answer is 50%. But this is wrong. To see why, consider the sample space S, below, for the "experiment" of having a family of two children.



Let A be the event of there being at least one girl. You met a woman, so A has occurred. Let B be the event of the two siblings including at least one boy. We seek the probability of B given that A has occurred. Looking at the above diagram, we see that B occurs in 2 out of the 3 equally likely outcomes in A, so the answer to the question is $p(B|A) = \frac{2}{3} = 66.\overline{6}\%$.

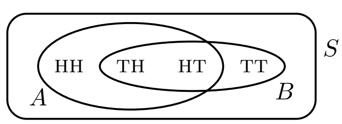
Alternatively, we can use Formula 2 from Fact 7.3 to get the answer as

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3} = 66.\overline{6}\%.$$ ✍

If the answer to Exercise 7.13 seems paradoxical and unexpected, then the next example will give an alternate way of thinking about it that is actually verifiable. It is exactly the same as the previous example, but in a different guise.

Example 7.14. Someone tells you they tossed a coin twice, and one toss was heads. What are the chances that the other toss was tails?

Solution. Below is the sample space for this experiment. Let A be the event of getting at least one head, and let B be the event of getting at least one tail.



Event A has occurred, so the probability that B will occur is $P(B|A) = 66.\overline{6}\%$, just as in the previous example.

This answer can be verified experimentally. Flip a coin twice, 100 times, and record the outcome for each of the 100 trials. Then among the outcomes that had at least one head, you will find that about 66% of them consist of a head *and* a tail.

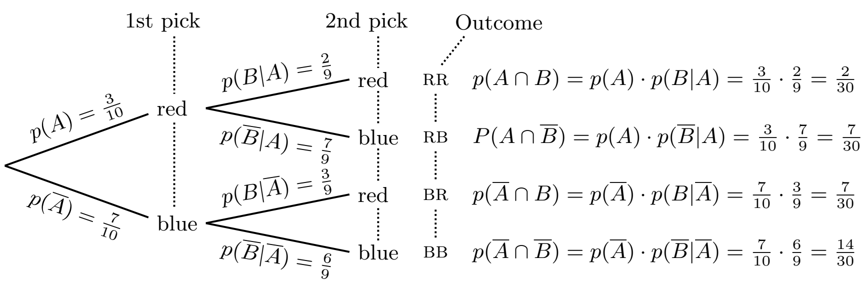
In the same way, if you meet a woman who has one sibling, there is a 66% chance that the sibling is a brother. ✍

Our next example involves an experiment with a sample space in which not all outcomes are equally likely.

Example 7.15. A jar contains 3 red balls and 7 blue balls. You reach in, pick a ball at random, and remove it. Then you randomly remove a second ball. Thus the sample space for this experiment $S = \{\text{RR, RB, BR, BB}\}$. Find the probability of each outcome in S.

Solution. Form the events A: *First pick is red,* and B: *Second pick is red.* Then we have $\overline{A}$: *First pick is blue,* and $\overline{B}$: *Second pick is blue.*

The probability of the first pick is red is $p(A) = \frac{3}{10}$, as there are 3 out of 10 red balls that we could have picked. Once this has happened, there are 9 balls left, two of which are red, so $p(B|A) = \frac{2}{9}$. So the probability that both picks are red is $p(\text{RR}) = p(A \cap B) = p(A) \cdot p(B|A) = \frac{3}{10} \cdot \frac{2}{9} = \frac{2}{30}$. This is tallied in the top branch of the following tree.



Likewise, $p(\text{RB}) = p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}|A)$. To find $p(\overline{B}|A)$, note that if A has occurred, then there are 9 balls left in the jar, and 7 of them are blue, so $p(A \cap \overline{B}) = \frac{7}{9}$. Thus $p(\text{RB}) = p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}|A) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$, and this is shown in the second-from-the-top branch of the tree.

Similar computations for the probabilities of the remaining two outcomes are shown on the bottom branches. Check that you understand them. From this tree we see that the probabilities of the various outcomes in S are

$$S = \left\{ \begin{array}{cccc} \text{RR}, & \text{RB}, & \text{BR}, & \text{BB} \\ 6.\overline{6}\% & 23.\overline{3}\% & 23.\overline{3}\% & 46.\overline{6}\% \end{array} \right\}$$

✍

If in the above Example 7.15, we had been asked for the probability of the event $E = \{\text{RB, BR}\}$ of the two picks being different colors, we would surmise that $p(E) = p(\text{RB}) + p(\text{BR}) = 23.\overline{3}\% + 23.\overline{3}\% = 46.\overline{6}\%$.

Further, notice that the probabilities of the individual outcomes in S add to 100%, so $P(S) = 100\%$, which makes sense because S is event of some outcome happening when the experiment is performed.

The next section is a further exploration of situations such as this one, in which not all outcomes in a sample space are equally likely.

Exercises for Section 7.3

1. The top card in a shuffled 52-card deck is a club. What is the probability that the bottom card is red?

2. The top card in a shuffled 52-card deck a heart. What is the probability that the bottom card is red?

3. A man has three children, and there are more girls than boys. What is the probability that his oldest child is a boy?

4. Toss a coin four times. Consider these events: A: *There were two heads and two tails*, and B: *The first toss was a tail.* Find $p(A)$, $p(B)$, $P(A \cap B)$, $p(A|B)$ and $p(B|A)$. Are events A and B independent?

5. Toss a coin four times. Consider these events: A: *The first two tosses are tails,* and B: *The last three tosses are tails.* Find $p(A)$, $p(B)$, $p(A \cap B)$, $p(A|B)$ and $p(B|A)$. Are events A and B independent?

6. Suppose $A, B \subseteq S$ are two events in the sample space S of some experiment. Suppose $p(A) = 60\%$, $p(B) = 80\%$ and $p(A|B) = 50\%$. Find $p(A \cap B)$, $p(A \cup B)$, $p(B|A)$ and $p\left(\overline{B}\right)$. Are A and B independent or dependent?

7. A box contains six tickets: [A] [A] [B] [B] [B] [E]. You take two tickets, one after the other. What is the probability that the first ticket is an A and the second is B?

8. A box contains six tickets: [A] [A] [B] [B] [B] [E]. You remove two tickets, one after the other. What is the probability that both tickets are vowels?

9. In a shuffled 52-card deck, what is the probability that the top card is red and the bottom card is a heart?

10. A card is drawn off a 52-card deck. Let A be the event "The card is a heart." Let B be the event "The card is a queen." Are these events independent or dependent?

11. Suppose A and B are events, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{6}$. Are A and B independent, dependent, or is there not enough information to say?

12. Suppose A and B are events, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cup B) = \frac{2}{3}$. Are A and B independent, dependent, or is there not enough information to say?

13. Say A and B are events with $P(A) = \frac{2}{3}$, $P(A|B) = \frac{3}{4}$, and $P(B|A) = \frac{1}{2}$. Find $p(B)$.

14. A box contains 2 red balls, 3 blue balls, and 1 green ball. You remove two balls, one after the other. Find the probability that both balls are red. Find the probability that both balls are blue. Find the probability that both balls have the same color.

15. A box contains 2 red balls, 3 black balls, and 4 white balls. One is removed, and then another is removed. What is the probability that no black balls were drawn?

16. Flip a coin 5 times. What is the probability of all 5 tosses being tails? If there were more tails than heads, then what is the probability that all 5 tosses were tails?

17. A coin is flipped 5 times, and there are more tails than heads. What is the probability that the first flip was a tail?

18. Suppose events A and B are independent, $p(A) = \frac{1}{3}$, and $p(A \cup B) = \frac{2}{3}$. Find $p(B)$.

19. A 5-card hand is dealt from a shuffled 52-card deck. Exactly 2 of the cards in the hand are hearts. Find the probability that all the cards in the hand are red.

20. A 5-card hand is dealt from a shuffled 52-card deck. Exactly 2 of the cards in the hand are red. Find the probability that all the cards in the hand are hearts.

7.4 Probability Distributions and Probability Trees

Except for Example 7.15 on the previous page, we have, until now, assumed that any two outcomes in a sample space are equally likely to occur. This is reasonable in many situations, such as tossing an unbiased coin or dice, or dealing a hand from a shuffled deck.

But in reality, things are not always so uniform. Suppose the spots of a dice are hollowed out, and when tossed it is more likely to land with a lighter side up (one with more spots). Toss the dice once. The probabilities of the six outcomes in the sample space might be something like this:

$$S = \left\{ \begin{array}{cccccc} \text{⚀}, & \text{⚁}, & \text{⚂}, & \text{⚃}, & \text{⚄}, & \text{⚅} \\ 15\% & 15\% & 16\% & 16\% & 18\% & 20\% \end{array} \right\}$$

(Of course it's unlikely the percentages would be whole numbers; this is just an illustration.) Note that the probabilities of all outcomes sum to 1:

$$p(\text{⚀}) + p(\text{⚁}) + p(\text{⚂}) + p(\text{⚃}) + p(\text{⚄}) + p(\text{⚅}) = 1,$$

because if tossed, the probability of its landing on one of its six faces is 100%. The probability of an event such as $E = \{\text{⚁}, \text{⚃}, \text{⚅}\}$ (*lands on even*) is

$$p(\text{⚁}) + p(\text{⚃}) + p(\text{⚅}) = 15 + 16 + 20 = 51\%.$$

Formula 7.1 does not apply here because the outcomes are not all equally likely. In fact it gives the incorrect probability $p(E) = \frac{|E|}{|S|} = \frac{3}{6} = 50\%$.

These ideas motivate the main definition of this section.

> **Definition 7.4.** For an experiment with sample space $S = \{x_1, x_2, \ldots, x_n\}$, a **probability distribution** is a function p that assigns to each outcome $x_i \in S$ a probability $p(x_i)$ with $0 \leq p(x_i) \leq 1$, and for which
>
> $$p(x_1) + p(x_2) + \cdots + p(x_n) = 1.$$
>
> The **probability** $p(E)$ of an event $E \subseteq S$ is the sum of the probabilities of the elements of E.

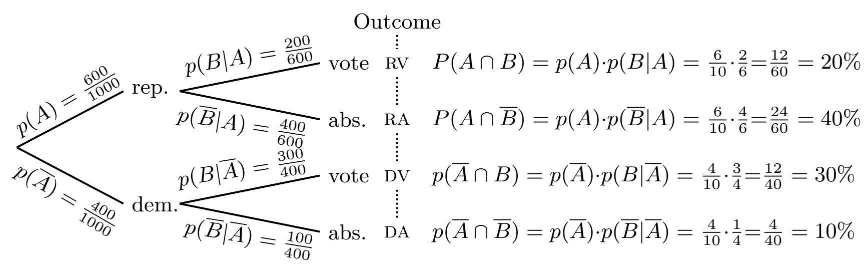
In the case where all outcomes are equally likely, any outcome $x_i \in S$ has probability $p(x_i) = \dfrac{1}{|S|}$. This is a probability distribution, by Definition 7.4. It is called the **uniform distribution** on S. For the uniform distribution we have the formula $p(E) = \dfrac{|E|}{|S|}$, but, as noted above, this may not hold for non-uniform probability distributions.

Example 7.16. A certain voting precinct has 1000 voters, 600 of whom are republicans and 400 of whom are democrats. In a recent election, 200 republicans voted and 300 democrats voted. You randomly select a member of the precinct and record whether they are republican or democrat, and whether or not they voted. Thus the sample space for the experiment is $S = \{\text{RV}, \text{RA}, \text{DV}, \text{DA}\}$, where RV means your selection was a republican who voted, whereas RA indicates a republican who abstained from voting, etc.

Find the probability distribution for S. Also, find the probability that your selection was a republican who voted or a democrat who didn't.

Solution. The chance that you picked a republican is $\frac{600}{1000} = 60\%$, and the chance you picked a democrat is $\frac{400}{1000} = 40\%$. If you picked a republican, the conditional probability that this person voted is $\frac{200}{600}$, and the conditional probability that they didn't vote is $\frac{400}{600}$. If you picked a democrat, the conditional probability that this person voted is $\frac{300}{400}$, and the conditional probability that they didn't vote is $\frac{100}{400}$.

Here is the probability tree, where $A = \{\text{RV}, \text{RA}\}$ is the event of picking a republican and $B = \{\text{RV}, \text{DV}\}$ is the event of picking a voter.



Thus the probability distribution is

$$S = \left\{ \begin{matrix} \text{RV}, & \text{RA}, & \text{DV}, & \text{DA} \\ 20\% & 40\% & 30\% & 10\% \end{matrix} \right\}$$

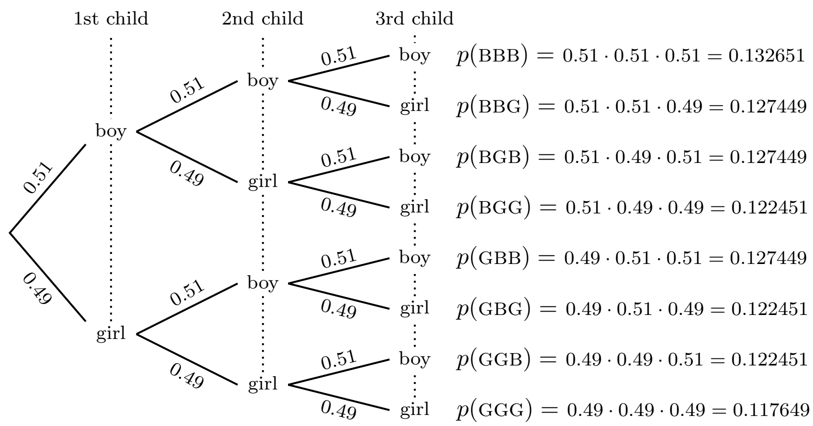
The probability that you picked a republican who voted or a democrat who didn't is $p\left(\{\text{RV}, \text{DA}\}\right) = p(\text{RV}) + p(\text{DA}) = 20\% + 10\% = 30\%$. ✍

At the beginning of Section 7.3 we calculated the probability that, for a family of three children, more girls than boys is just as likely as the oldest child being a boy. This was based on the assumption that there is a 50-50 chance of each child being a boy or a girl.

In reality, there is about a 51% chance of a child being born a boy, versus 49% for a girl. (Though the mortality rate for boys is higher, so this statistic is somewhat equalized in adulthood.) Let's revisit our question.

Example 7.17. Assume that there is 51% chance of a child being born a boy, versus 49% of being a girl. For a family of three children, consider events A: *There are more girls than boys,* and B: *The oldest child is a boy.* Find $p(A)$ and $p(B)$.

Solution. The sample space is $S = \{\text{BBB}, \text{BBG}, \text{BGB}, \text{BGG}, \text{GBB}, \text{GBG}, \text{GGB}, \text{GGG}\}$. The following probability tree computes the probability of each outcome. (We assume that gender of births are independent, that is, the gender of one child does not influence the gender of the next child born.)



The probability $p(A)$ of more girls than boys is

$$p(\{\text{BGG, GBG, GGB, GGG}\}) = p(\text{BGG}) + p(\text{GBG}) + p(\text{GBG}) + p(\text{GGG})$$
$$= 0.122451 + 0.122451 + 0.122451 + 0.117649 \approx \mathbf{48.5\%}.$$

The probability $p(B)$ of the oldest being a boy is

$$p(\{\text{BBB, BBG, BGB, BGG}\}) = p(\text{BGG}) + p(\text{GBG}) + p(\text{GBG}) + p(\text{GGG})$$
$$= 0.132651 + 127449 + 127449 + 0.122451 \approx \mathbf{51\%}. \quad ✍$$

If p is a probability distribution on a sample space S, then Definition 7.4 implies $p(S) = 1$ (because the probabilities of the elements of S sum to 1). Also, for mutually exclusive events $A, B \subseteq S$, we have $p(A \cup B) = p(A) + p(B)$, because Definition 7.4 says $p(A \cup B)$ is the sum probabilities of elements of A, plus those for B. And $S = A \cup \overline{A}$, so $1 = p(S) = p(A \cup \overline{A}) = p(A) + p(\overline{A})$. This implies $p(A) = 1 - p(\overline{A})$ and $p(\overline{A}) = 1 - p(A)$. Therefore the formulas 2, 3 and 4 of Fact 7.2 (page 170) hold for arbitrary probability distributions, even though we derived them earlier only for uniform distributions.

Similar reasoning gives the following facts, which can be taken as a summary of all probability formulas in this chapter. It may be useful for working this sections's exercises. (For completeness, it also gives *Bayes' formula*, which comes in the *next* section, and will *not* be needed on this section's exercises.)

Discrete Probability Summary

An **experiment** is an act that results in one of finitely many outcomes. The **sample space** $S = \{x_1, \dots, x_n\}$ of an experiment is the set of all possible outcomes. An **event** is a subset $E \subseteq S$.

If $A, B \subseteq S$ are arbitrary arbitrary events, then

- $E = A \cup B$ is the event "A *occurs* **or** B *occurs*,"
- $E = A \cap B$ is the event "A *occurs* **and** B *occurs*,"
- $E = \overline{A}$ is the event "A **does not** *occur*."

A **probability distribution** is a function p, assigning to each outcome $x_i \in S$ a probability $p(x_i)$ with $0 \le p(x_i) \le 1$, and $p(x_1) + p(x_2) + \cdots + p(x_n) = 1$. The **probability** $p(E)$ of an event $E \subseteq S$ is the sum of the probabilities of the elements of E. Thus $p(S) = 1$ and $p(\emptyset) = 0$.

If the outcomes in S are all equally likely to occur, then $p(x_i) = \frac{1}{|S|}$ for each $x_i \in S$, and p is called the **uniform distribution**. For an event E,

1. $p(E) = \dfrac{|E|}{|S|}$ .... if p is the uniform distribution.

Events A and B are **mutually exclusive** if $A \cap B = \emptyset$, meaning $p(A \cap B) = p(\emptyset) = 0$, that is, A and B cannot both happen at the same time. In general:

2. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

3. $p(A \cup B) = p(A) + p(B)$ .... if A and B are mutually exclusive

4. $p(\overline{A}) = 1 - p(A)$

5. $p(A) = 1 - p(\overline{A})$.

The **conditional probability** of A given B is denoted $p(A \mid B)$. It is the probability of A, given that B has occurred. Events A and B are **independent** if $p(A \mid B) = p(A)$ and $p(B \mid A) = p(B)$, that is, if one happening does not change the probability that the other will happen.

6. $p(A \mid B) = \dfrac{p(A \cap B)}{p(B)}$

7. $p(B \mid A) = \dfrac{p(A \cap B)}{p(A)}$

8. $p(A \cap B) = p(A \mid B) \cdot p(B) = p(A) \cdot p(B \mid A)$

9. $p(A \cap B) = p(A) \cdot p(B)$ .... if A and B are independent

Bayes' Formula (Section 7.5) Suppose a sample space S for an experiment is a union $S = S_1 \cup S_2$, with $S_1 \cap S_2 = \emptyset$. If $E \subseteq S$ is an event, then

10. $p(S_1 \mid E) = \dfrac{p(S_1) \cdot p(E \mid S_1)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}$

11. $p(S_2 \mid E) = \dfrac{p(S_2) \cdot p(E \mid S_2)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}$.

Exercises for Section 7.4

1. A shuffled standard deck of cards is missing the ace of heats. You draw one card from the deck and observe its suit. The sample space for this experiment is $S = \{\heartsuit, \clubsuit, \diamondsuit, \spadesuit\}$. Find the probability distribution on S.

2. A weighted coin is twice as likely to land on tails than heads. Find the probability distribution on the sample space for tossing the coin once. Then find the probability distribution on the sample space for tossing it twice.

3. A weighted coin is 1.5 times as likely to land on heads than tails. Find the probability distribution for the experiment of tossing the coin once. If you toss it three times, what are the chances of getting all heads? All tails? What are the chances of getting more heads than tails?

4. An unfair dice is equally likely to land on 1, 2, 3, 4, and 5, but it is twice as likely to land on on 6 than on any other number. Find the probability distribution for tossing the dice once. If you toss it twice, what are the chances that both tosses are 6. What are the chances that at least one toss is a 6?

5. There is a 40% chance of rain on Saturday and a 25% chance of rain on Sunday. What is the probability that it will rain on at least one day of the weekend? (You may assume that the events "Rain on Saturday" and "Rain on Sunday" are independent events.)

6. There is an 80% chance that there will be rain over the weekend, and a 50% chance of rain on Saturday. What is the chance of rain on Sunday? (You may assume that the events "Rain on Saturday" and "Rain on Sunday" are independent events.)

7. A box contains 7 red balls and 5 green balls. You reach in and remove two balls, one after the other. Complete the probability tree.

8. A box contains 8 red balls and 5 green balls. You reach in and remove two balls, one after the other. Complete the probability tree.

9. A box contains 7 red balls, 5 green balls and 1 blue ball. You reach in and remove two balls, one after the other. Complete the probability tree.

10. A box contains 8 red balls, 4 green balls and 1 blue ball. You reach in and remove two balls, one after the other. Complete the probability tree.

11. A club consists of 60 men and 40 women. To fairly choose a president and a secretary, names of all members are put into a hat and two names are drawn. The first name drawn is the president, and the second name drawn is the secretary. What is the probability that the president and the secretary have the same gender?

12. At a certain college, 40% of the students are male, and 60% are female. Also, 20% of the males are smokers, and 10% of the females are smokers. A student is chosen at random. What is the probability that the student is a male nonsmoker?

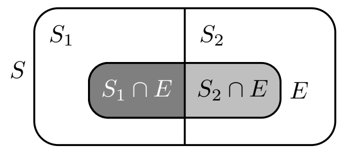
13. At a certain college, 30% of the students are freshmen. Also, 80% of the freshmen live on campus, while only 60% of the non-freshman students live on campus. A student is chosen at random. What is the probability that the student is a freshman who lives off campus?

14. Suppose events A and B are both independent *and* mutually exclusive. What can you say about $p(A)$ and $p(B)$?

15. An unfair coin is twice as likely to land on heads as tails.
If you toss it 10 times what are the chances that you will get exactly 6 heads?
If you toss it n times what are the chances that you will get exactly k heads?

7.5 Bayes' Formula

We are going to learn one final probability formula, *Bayes' formula*, named for its discoverer, Thomas Bayes (1702–1761). His formula gives an answer to the following question: Suppose a sample space S for an experiment is a union $S = S_1 \cup S_2$, with $S_1 \cap S_2 = \emptyset$, and $E \subseteq S$ is an event. If E occurs, then what is the probability that S_1 has occurred? That is, what is $p(S_1 \mid E)$?



A short computation gives the answer.

$$\begin{aligned}
p(S_1 \mid E) &= \frac{p(S_1 \cap E)}{p(E)} && \text{by formula 5 on page 186} \\
&= \frac{p(S_1)\cdot p(E \mid S_1)}{p(E)} && \text{by formula 7 on page 186} \\
&= \frac{p(S_1)\cdot p(E \mid S_1)}{p\big(\,(S_1 \cap E) \cup (S_2 \cap E)\,\big)} && \text{as } E = (S_1 \cap E) \cup (S_2 \cap E) \\
&= \frac{p(S_1)\cdot p(E \mid S_1)}{p(S_1 \cap E) + p(S_2 \cap E)} && \text{because } S_1 \cap E \text{ and } S_2 \cap E \text{ are mutually exclusive} \\
&= \frac{p(S_1)\cdot p(E \mid S_1)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2)}. && \text{by formula 7 on page 186}
\end{aligned}$$

The same steps give $p(S_2 \mid E) = \dfrac{p(S_2)\cdot p(E \mid S_2)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2)}$.

Fact 7.4. Bayes' Formula
Suppose a sample space S for an experiment is a union $S = S_1 \cup S_2$, with $S_1 \cap S_2 = \emptyset$. Suppose also that $E \subseteq S$ is an event.

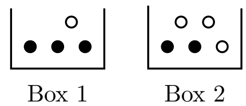
Then $p(S_1 \mid E) = \dfrac{p(S_1)\cdot p(E \mid S_1)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2)}$

and $p(S_2 \mid E) = \dfrac{p(S_2)\cdot p(E \mid S_2)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2)}$.

Though we will not use it here, we mention that Bayes' formula extends to situations in which S decomposes into more than two parts. If $S = S_1 \cup S_2 \cup \cdots \cup S_n$, and $S_i \cap S_j = \emptyset$ whenever $1 \le i < j \le n$, then for any S_i,

$$p(S_i \mid E) = \frac{p(S_i)\cdot p(E \mid S_i)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2) + \cdots + p(S_n)\cdot p(E \mid S_n)}. \quad (7.1)$$

Example 7.18. There are two boxes. Box 1 contains three black balls and one white ball. Box 2 contains two black balls and three white balls.



Someone chooses a box at random and then randomly takes a ball from it. The ball is white. What is the probability that the ball is from Box 1?

Solution. The sample space is $S = \{\,1\text{B},\ 1\text{W},\ 2\text{B},\ 2\text{W}\,\}$, where the number refers to the box the selected ball came from, and the letter designates whether the ball is black or white.

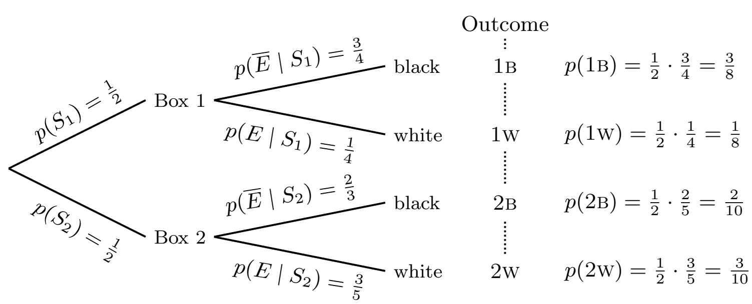
Let $S_1 = \{1\text{B},\ 1\text{W}\}$ be the event S_1 : *The ball came from Box* 1.
Let $S_2 = \{2\text{B},\ 2\text{W}\}$ be the event S_2 : *The ball came from Box* 2.
Let $E = \{1\text{W},\ 2\text{W}\}$ be the event E : *The ball is white.*

The answer to the question is thus $p(S_1 \mid E)$. As $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, Bayes' formula applies, and it gives

$$\begin{aligned} p(S_1 \mid E) &= \frac{p(S_1)\cdot p(E \mid S_1)}{p(S_1)\cdot p(E \mid S_1) + p(S_2)\cdot p(E \mid S_2)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{4}}{\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{5}} = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{3}{10}} = \frac{\frac{10}{80}}{\frac{34}{80}} = \frac{10}{34} \approx 29.4\%. \end{aligned}$$

So there's a 29.4% chance that the selected white ball came from Box 1. ✍

For full disclosure, let it be known that we could bypass Bayes' formula and solve this problem with a probability tree. Note that $\overline{E} = \{1\text{B},\ 2\text{B}\}$ is the event of a black ball being chosen. Consider the following probability tree.



Applying Formula 5 from page 186 to these figures, our answer is

$$p(S_1|E) = \frac{p(S_1\cap E)}{p(E)} = \frac{p\big(\{1\text{W}\}\big)}{p\big(\{1\text{W},\ 2\text{W}\}\big)} = \frac{p(1\text{W})}{p(1\text{W})+p(2\text{W})} = \frac{\frac{1}{8}}{\frac{1}{8}+\frac{3}{10}} = \frac{10}{34} \approx 29.4\%.$$

You may find that the next example plays out in a very unexpected way.

Example 7.19. A certain disease occurs in only 1% of the population. A pharmaceutical company develops a test for this disease. They make the following claims about their test's accuracy in the event a subject is tested:

If you have the disease, then there is a 99% chance you will test positive.
If you don't have the disease, there is a 99% chance you will test negative.

You take the test and you test positive. Assuming that the pharmaceutical company's claims are accurate, what is the chance that you have the disease?

Solution: Given the data, you may suspect that there is a high probability that you have the disease. However, this is not so, and Baye's formula can give the exact answer. We can set up the problem as an experiment: You take the test. There are four possible outcomes:

HP (you Have the disease and test Positive);
HN (you Have the disease and test Negative);
DP (you Don't have the disease and test Positive);
DN (you Don't have the disease and test Negative);

Consequently the sample space is

$$S = \{ \underbrace{HP,\ HN}_{S_1},\ \underbrace{DP,\ DN}_{S_2} \},$$

where $S_1 = \{HP, HN\}$ is the event of having the disease and $S_2 = \{DP, DN\}$ is the event of not having it. Further, $E = \{HP, DP\}$ is the event of testing positive for the disease.

We seek the probability that you have the disease given that you tested positive, that is, we seek $p(S_1 \mid E)$. Note $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, so we have the set-up for Baye's formula. The formula says our answer will be

$$p(S_1 \mid E) = \frac{p(S_1) \cdot p(E \mid S_1)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)}.$$

Note that, $p(S_1) = 0.01$ because only 1% of the population has the disease. Similarly, $p(S_2) = 0.99$ because 99% of the population does not have it. The pharmaceutical company said that there is a 99% chance that you will test positive if you have the disease, which is to say $p(E \mid S_1) = 99\%$. They also said that there is a 99% chance you will test negative if you don't have the disease, and from this we infer that there is a 1% chance that you will test *positive* if you don't have the disease; this means $p(E \mid S_2) = 1\%$. Thus

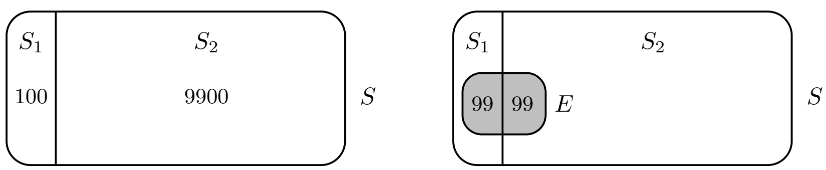
$$p(S_1 \mid E) = \frac{0.01 \cdot 0.99}{0.01 \cdot 0.99 + 0.99 \cdot 0.01} = \frac{0.0099}{0.0099 + 0.0099} = 0.5 = 50\%.$$

So if you test positive, there is a **50%** chance that you have the disease. ✍

Example 7.19 is a cautionary tale about the risks of jumping to a seemingly reasonable (but incorrect) conclusion based on a hasty analysis of data. If you tested positive, then—given the claims about the test's accuracy—it seems as if you would be very likely to have the disease. Yet the chance is only 50%. This may seem somehow paradoxical.

To see that the answer really makes sense, lets analyze the problem in a different way, as a scaled-down thought experiment. Imagine that we have a population of 10,000 people, and they all get tested for the disease. The experiment will involve picking a person at random, so the sample space S is the set of all people. Let $S_1 \subseteq S$ be the set of people that have the disease, and let $S_2 \subseteq S$ be the set of people that do not have it.

Only 1% of S has the disease, and thus $|S_1| = 0.01 \cdot 10,000 = 100$ people. And because 99% of the population is disease-free, we know $|S_2| = 0.99 \cdot 10,000 = 9900$ people. We record this in the Venn diagram below, left. (For clarity, the diagram is not quite to scale, as technically S_1 should be 1% of the area enclosed by S, a tiny sliver.)



Next, let E be the set of people who tested positive for the disease. The part of E that overlaps S_1 consists of the people who have the disease and tested positive for it. The pharmaceutical company stated that 99% of the 100 people in S_1 (who have the disease) will test positive, so $|S_1 \cap E| = 99$. See the diagram above, right. On the other hand, the part of E that overlaps S_2 consists of the people who *do not have* the disease and tested positive for it. Because 99% of people who do not have the disease will test negative, only 1% of them will test positive. This means that the part of E that overlaps S_2 contains 1% of the 9900 people in S_2, which is 99 people.

Consequently, the set E of those who tested positive consists of 99 people who have the disease and 99 people who do not have it. So if we randomly select a person who tested positive, then that person is just as likely to have the disease (i.e., to be in S_1) as not have it (i.e., to be in S_2).

Thus the unexpected answer to Example 7.19 comes from the fact that the disease is very rare, so S_1 is a tiny fraction of S. It happens that 99% of S_1 is equal to 1% of S_2, so the number of people who tested positive and have the disease equals the number who tested positive and do not have it.

So the claims about the test's accuracy are misleading: if you tested positive, then there is only a 50% that you actually have the disease.

Exercises for Section 7.5

1. At a certain college, 40% of the students are male, and 60% are female. Also, 20% of the males are transfer students, and 10% of the females are transfer students. A student is chosen at random. If the student is a transfer student, what is the probability that the student is female?

2. At a certain college, 30% of the students are freshmen. Also, 80% of the freshmen live on campus, while only 60% of the non-freshman students live on campus. A student is chosen at random. If the student lives on campus, what is the probability that the student is a freshman?

3. A jar contains 4 red balls and 5 white balls. A random ball is removed, and then another is removed. If the second ball was red, what is the probability that the first ball was red?

4. University of Alpha has a population of 10,000 students, 51% of them men, and 49% women. University of Beta has 7000 students, 40% men, and 60% women. A random student is selected from the two populations. If the student is male, what is the probability he is from University of Alpha?

5. A retail store obtains battery packs from two distributers, Distributer A and Distributer B. On a particular day, 70% of the packs in the store are from Distributer A, and 30% are from Distributer B. It happens that 1% of the packs from Distributer A contain a defective battery, and 2% of the packs from Distributer B contain a defective battery. A consumer buys a pack that contains a defective battery. What is the probability that it came from Distributer B?

6. One week printer prints copies of a particular book at a rate of 1000 copies per day. Monday through Wednesday it prints 600 paperback editions and 400 hardcover editions. On Thursday and Friday it prints 800 paperbacks and 200 hardcovers. If you purchase a paperback edition that was printed that week, what is the probability that it was printed on Thursday or Friday.

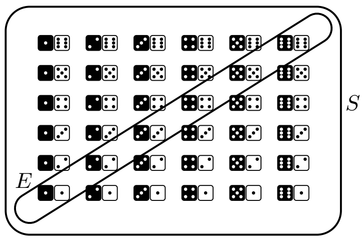
7. An alternate form of Bayes' formula is $p(A|B) = \dfrac{p(B|A) \cdot p(A)}{p(B)}$, provided $p(B) \neq 0$. Derive this formula in three steps using formulas 1, 2 and 3 of Fact 7.3.

8. In the formula from Exercise 7 above, set $S_1 = A$, $S_2 = \overline{A}$ and $E = B$ to derive Bayes' formula for $P(S_1|E)$ as stated in Fact 7.4.

Solutions for Chapter 7

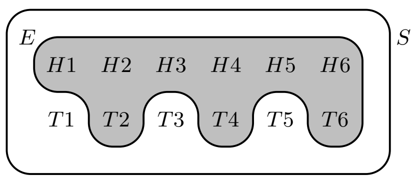
Section 7.1

1. You have two dice, a white one and a black one. You toss both of them together. Find the probability that they both show the same number.



$$p(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6} = \boxed{16.\bar{6}\%}$$

3. You toss a coin and then roll a dice. You win \$1 if the the coin is heads or the dice is even. What are your chances of winning?



$$p(E) = \frac{|E|}{|S|} = \frac{9}{12} = \boxed{75\%}$$

5. A card is randomly selected from a deck of 52 cards. What is the chance that the card is red or a king?

Solution: The sample space S is the set of 52 cards. The experiment is drawing one card. The event $E \subseteq S$ is the set of the cards that are red or kings. This is the set of 26 red cards, plus the king of spades and the king of clubs. Therefore $|E| = 28$, and $p(E) = \frac{|E|}{|S|} = \frac{28}{52} \approx 53.8\%$.

7. Toss a dice 5 times in a row. What is the probability that you don't get any ⚅'s?

Solution: The sample space S is the set of all length-5 lists (repetition allowed) whose entries are the numbers 1, 2, 3, 4, 5, 6. There are $6^5 = 7776$ such lists, so $|S| = 7776$. The event E consists of those lists in S that do not contain a 6. There are $5^5 = 3125$ of them, so $p(E) = \frac{|E|}{|S|} = \frac{3125}{7776} \approx 40.187\%$.

9. Toss a dice 5 times in a row. What is the probability that you will get the same number on each roll? (i.e. ⚀⚀⚀⚀⚀ or ⚁⚁⚁⚁⚁, etc.)

Solution: The sample space S is the set of all length-5 lists (repetition allowed) whose entries are the numbers 1, 2, 3, 4, 5, 6. There are $6^5 = 7776$ such lists, so $|S| = 7776$. Note that $E = \{11111, 22222, 33333, 44444, 55555, 66666\}$, so $|E| = 6$, and $p(E) = \frac{|E|}{|S|} = \frac{6}{7776} \approx 0.077\%$.

11. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that they show different numbers?

Solution: The sample space S is shown in Example 7.1 on page 165. You can see that $|S| = 36$ and the event E of the two dice showing different numbers has cardinality

30, so $p(E) = \frac{|E|}{|S|} = \frac{30}{36} = 83.\overline{3}\%$.

13. You have a pair of dice, a white one and a black one. Toss them both. What is the probability that both show even numbers?

Solution: The sample space S is shown in Example 7.1 on page 165. Note that $E = \{$⚁⚁, ⚁⚃, ⚁⚅, ⚃⚁, ⚃⚃, ⚃⚅, ⚅⚁, ⚅⚃, ⚅⚅$\}$. Thus $p(E) = \frac{|E|}{|S|} = \frac{9}{36} = 25\%$.

15. Toss a coin 8 times. Find the probability that the first and last tosses are heads.

Solution: The sample space S is the set of all length-8 lists made from the two symbols H and T. Thus $|S| = 2^8$. The event E of the first and last tosses being heads consists of all those outcomes in S that have the form $(\text{H}, \square, \square, \square, \square, \square, \square, \text{H})$ where there are two choices for each box. Thus $|E| = 2^6$, and so $p(E) = \frac{|E|}{|S|} = \frac{2^6}{2^8} = \frac{1}{2^2} = 25\%$.

17. Five cards are dealt from a shuffled 52-card deck. What is the probability of getting three red cards and two clubs?

Solution: The sample space S is the set of all possible 5-card hands that can be made from the 52 cards in the deck, so $|S| = \binom{52}{5} = 2,598,960$. There are $\binom{26}{3}$ ways to get 3 red cards and $\binom{13}{2}$ ways to get 2 clubs, so by the multiplication principle there are $\binom{26}{3}\binom{13}{2} = 202,800$ different 5-card hands that have 3 red cards and 2 clubs. Therefore $p(E) = \frac{|E|}{|S|} = \frac{202{,}800}{2{,}598{,}960} \approx 7.803\%$.

19. Alice and Bob each randomly pick an integer from 0 to 9. What is the probability they pick the same number? What is the probability they pick different numbers?

Solution: Lets put $S = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq a, b \leq 9\} = \{(0,0), (0,1), (0,2), \ldots, (9,9)\}$ where a ordered pair (a, b) means Alice picked a and Bob picked b. Then $S = 10 \cdot 10 = 100$. The event of their both picking the same number is $E = \{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9)\}$, so $|E| = 10$. Then the probability of their picking the same number is $p(E) = \frac{|E|}{|S|} = \frac{10}{100} = 10\%$. The event of their picking different numbers is $\overline{E} = S - E$, so the probability of their picking different numbers is $p(E) = \frac{|\overline{E}|}{|S|} = \frac{90}{100} = 90\%$.

21. What is the probability that a 5-card hand dealt off a shuffled 52-card deck does not contain an ace?

Solution: The sample space S is the set of all possible 5-card hands that can be made from the 52 cards in the deck, so $|S| = \binom{52}{5} = 2,598,960$. To make a 5-card hand that contains no ace, we have to choose 5 cards from the 48 non-ace cards, so there are $\binom{48}{5} = 1,712,304$ hands that contain no aces. Thus the probability of the event E of no aces in the hand is $p(E) = \frac{|E|}{|S|} = \frac{1{,}712{,}304}{2{,}598{,}960} \approx 65.88\%$.

Section 7.2

1. A card is taken off the top of a shuffled 52-card deck. What is the probability that it is black or an ace?

Solution: Let A be the event of the card being an ace, and let B be the event of its being black. Then $p(A) = \frac{4}{52}$, and $p(B) = \frac{26}{52}$. The event $A \cap B$ is the event of the card being either the ace of spades or the ace of clubs. Thus $|A \cap B| = 2$, and $p(A \cap B) = \frac{2}{52}$. The answer we seek is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} \approx 53.8\%$.

3. What is the probability that a 5-card hand dealt off a shuffled 52-card deck contains at least one red card?

Solution: The sample space S is the set of all 5-card hands, so $|S| = \binom{52}{5} = 2,598,960$. Let E be the event of a 5-card hand without any red cards. Then $|E| = \binom{26}{5} = 65,780$ (choose 5 cards from the 26 black cards). Note that the complement $\overline{E}$ is the event of at least one red card, so the answer we seek is $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{65,780}{2,598,960} \approx 97.46\%$.

5. You toss a fair coin 8 times. What is the probability that you do not get 4 heads?

Solution: The sample space S is the set of all length-8 lists made from the symbols H and T. Thus $|S| = 2^8$. Now let E be the event of getting exactly 4 heads, so $|E| = \binom{8}{4} = 70$. (Choose 4 of 8 positions for H and fill the rest with T.) Then $\overline{E}$ is the event of not getting four heads. Our answer is then $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{70}{2^8} \approx 72.65\%$.

7. Two cards are dealt off a shuffled 52-card deck.What is the probability that the cards are both red or both aces?

Solution: The sample space S consists of all possible 2-card hands, so $|S| = \binom{52}{2} = 1326$. Let A be the event of both cards being aces, so $|A| = \binom{4}{2} = 6$. Let B be the event that both cards are red, so $|B| = \binom{26}{2} = 325$. Then the event $A \cap B$ consists on only one hand, namely the 2-card hand consisting of the ace of hearts and the ace of diamonds. The answer to our question is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \frac{6}{1326} + \frac{325}{1326} - \frac{1}{1326} \approx 24.88\%$.

9. A dice is tossed six times. You win \$1 if the first toss is a five or the last toss is even. What are your chances of winning?

Solution: The sample space S is the set of all length-6 lists made from the symbols 1, 2, 3, 4, 5 and 6. Thus $|S| = 6^6$. Let A be the event of the first toss being a five. By the multiplication principle, $|A| = 6^5$. Let B be the event of the last toss being even, that is, 2, 4 or 6. Then $|B| = 6^5 \cdot 3$. Note that $A \cap B$ is the set of all lists in S whose first entry is 5 and whose last entry is even. By the multiplication principle, $|A \cap B| = 6^4 \cdot 3$. The probability that the first toss is a five and the last is even is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \frac{6^5}{6^6} + \frac{6^5 \cdot 3}{6^6} - \frac{6^4 \cdot 3}{6^6} = \frac{1}{6} + \frac{3}{6} - \frac{3}{36} = \frac{21}{36} \approx 58.33\%$. So you have a reasonably good chance of winning.

11. A dice is rolled 5 times. Find the probability that not all of the tosses are even.

Solution: Think of the sample space as being the set of all length-5 lists made from the numbers 1, 2, 3, 4, 5 and 6, where the first entry is the result of the first roll, the second entry is the result of the second roll, etc. Thus $|S| = 6^5$. Now let E be the event of all rolls being even. Then E is the set of all length-5 lists made from the numbers 2, 4 and 6, so $|E| = 3^5$. We are interested in the probability of the event of not all rolls being even, that is, the probability of $\overline{E}$. Thus our answer is $p(\overline{E}) = 1 - p(E) = 1 - \frac{|E|}{|S|} = 1 - \frac{3^5}{6^5} \approx 96.875\%$.

13. Two cards are dealt off a well-shuffled deck. You win \$1 if the two cards are of different suits. Find the probability of your winning.

Solution: The sample space S is the set of all possible 2-card hands, so $|S| = \binom{52}{2} = 1326$. There are $\binom{13}{2} = 78$ 2-card hands with both cards hearts, and similarly 78 hands with both cards diamonds, 78 hands with both cards clubs, and 78 hands with both cards spades. By the addition principle there are $78 + 78 + 78 + 78 = 312$ hands in S for which both cards are of the same suit. So there are $|S| - 312 = 1014$ hands in S for which the cards are of different suits. Thus the probability of the two cards being the same suit is $\frac{1014}{|S|} = \frac{1014}{1326} \approx 76.47\%$.

15. A coin is tossed 5 times. What is the probability that the first toss is a head or exactly 2 out of the five tosses are heads?

Solution: The sample space S is the set of length-5 lists made from the symbols H and T, so $|S| = 2^5 = 32$. The event $A \subseteq S$ of the first toss being a head is the set of all lists in S of form H□□□□, so $|A| = 2^4 = 16$. The event B of exactly two heads has cardinality $|B| = \binom{5}{2} = 10$. (Choose two of 5 positions for H, and fill the rest with T's.) Finally, $A \cap B$ is the set of lists in S whose first entry is H and exactly one of the four remaining entries is an H, so $|A \cap B| = 4$. So the probability of the first toss being a head or exactly two tosses being heads is $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|} = \frac{16}{32} + \frac{10}{32} - \frac{4}{32} = \frac{22}{32} = 68.75\%$.

17. In a shuffled 52-card deck, what is the probability that neither the top nor bottom card is a heart?

Solution: Regard the sample space as the set of 2-element lists (no repetition) whose entries are the cards in the deck. The first entry represents the top card and the second entry represents the bottom card. Then $|S| = 52 \cdot 51 = 2652$. Now let E be the event that neither the top nor bottom card is a heart. So E consists of those lists in S for which neither entry is a heart. As there are 39 non-heart cards, $|E| = 39 \cdot 38 = 1482$. The answer is thus $p(E) = \frac{|E|}{|S|} = \frac{1482}{2652} \approx 55.88\%$.

19. A bag contains 20 red marbles, 20 green marbles and 20 blue marbles. You reach in and grab 15 marbles. What is the probability that they are all the same color?

Solution: An outcome for this experiment is a 15-element multiset made from the symbols $\{$R, G, B$\}$. Thus the sample space S is the set of all such multisets. Encode the elements of S as stars and bars, so a typical element of S is a list of length $15 + 2 = 17$, made from 15 stars and 2 bars.

$$\underbrace{* * \cdots * *}_{* \text{ for each R}} | \underbrace{* * \cdots * *}_{* \text{ for each G}} | \underbrace{* * \cdots * *}_{* \text{ for each B}}$$

Then $|S| = \binom{17}{2} = 136$. Also the event E of all balls being the same color is $E = \{ *************** ||,\ |***************|,\ ||*************** \}$, so $|E| = 3$. Finally, $p(E) = \frac{|E|}{|S|} = \frac{3}{136} \approx 2.2\%$.

Section 7.3

1. The top card in a shuffled 52-card deck is a club. What is the probability that the bottom card is red?

Of the bottom 51 cards, we know that there are 26 red cards and 25 black cards (the top card is black). Thus the chance that the card at the very bottom is red is $26/51 \approx 50.98\%$. Note that we have just computed a conditional probability. Consider events A: *Top card is a club,* and B: *Bottom card is red.* The answer to the question is $p(B|A) = 26/51$.

3. A man has three children, and there are more girls than boys. What is the probability that his oldest child is a boy?

Solution: The event of having three children, more girls than boys is $A = \{GGG, GGB, GBG, BGG\}$. Only one of the four outcomes in A has the oldest child as a boy, So the probability that his oldest child is a boy is $\dfrac{1}{4} = \boxed{\mathbf{25\%.}}$

Note: We got this answer without using the formula for conditional probability. To use the formula, notice that $|S| = 8$ and the event of the oldest child being a boy is $B = \{BBB, BGB, BBG, BGG\}$. Then the answer to the question is $P(B|A) = \frac{p(B\cap A)}{p(A)} = \frac{p(\{BGG\})}{p(\{GGG,GGB,GBG,BGG\})} = \frac{1/8}{4/8} = 1/4 = \boxed{\mathbf{25\%.}}$

5. Toss a coin four times. Consider events: A: *The first two tosses are tails,* and B: *The last three tosses are tails.* Find $p(A)$, $P(B)$, $p(A\cap B)$, $p(A|B)$ and $p(B|A)$. Are A and B independent?

Solution: The sample space S is the set of length-4 strings made from the letters H and T, so $|S| = 2^4 = 16$. Note that $A = \{TTTT, TTTH, TTHT, TTHH\}$, so $|A| = 4$. Also $B = \{TTTT, HTTT\}$, so $|B| = 2$. Finally, $A \cap B = \{TTTT\}$, so $|A \cap B| = 1$. Then $p(A) = |A|/|S| = 1/4$ and $p(B) = |B|/|S| = 1/8$. Also $p(A\cap B) = |A\cap B|/|S| = 1/16$. Then $p(A|B) = p(A\cap B)/p(B) = (1/16)/(1/8) = 1/2$ and $p(B|A) = p(A\cap B)/p(A) = (1/16)/(1/4) = 1/4$. Because $p(A) \neq p(A|B)$, events A and B are not independent.

7. A box contains six tickets: $\boxed{\boxed{\text{A}}\,\boxed{\text{A}}\,\boxed{\text{B}}\,\boxed{\text{B}}\,\boxed{\text{B}}\,\boxed{\text{E}}}$. You remove two tickets, one after the other. What is the probability that the first ticket is an A and the second is a B?

Solution: Let A be the event of the first draw being an A and let B be the event of the second draw being a B. With the help of Fact 7.3, the answer to this question is $p(A\cap B) = p(A)\cdot P(B|A) = \frac{2}{6}\cdot\frac{2}{5} = \frac{2}{15} = 13.\overline{3}\%$.

9. In a shuffled 52-card deck, what is the probability that the top card is red and the bottom card is a heart?

Solution: Let A be the event of the top card being red, and let B be the event of the bottom card being a heart. With the help of Fact 7.3, the answer to this question is $p(A\cap B) = p(B)\cdot P(A|B) = \frac{13}{52}\cdot\frac{25}{51} = \frac{25}{204} \approx 12.25\%$. (Notice that if you use the formula $p(A\cap B) = p(A)\cdot p(B|A)$, then the problem is somewhat harder to think about, because if the top card is red, it may or may not be a heart.)

11. Suppose A and B are events, and $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A\cap B) = \frac{1}{6}$. Are A and B independent, dependent, or is there not enough information to say for sure?

Solution: Using the information given, and Fact 7.3, we get $\frac{1}{6} = p(A\cap B) = p(A)\cdot p(B|A) = \frac{1}{2}p(B|A)$, which yields $p(B|A) = \frac{1}{3}$, so $p(B|A) = p(B)$. Also, $\frac{1}{6} = p(A\cap B) = p(B)\cdot p(A|B) = \frac{1}{3}p(A|B)$, which yields $p(A|B) = \frac{1}{2}$, so $p(A) = p(A|B)$. This means A and B are independent.

13. Say A and B are events with $P(A) = \frac{2}{3}$, $P(A|B) = \frac{3}{4}$, and $P(B|A) = \frac{1}{2}$. Find $p(B)$.

Solution: Fact 7.3 says $p(A)\cdot p(B|A) = p(A\cap B) = p(B)\cdot p(A|B)$. Plugging in the given information, this becomes $\frac{2}{3}\cdot\frac{1}{2} = p(B)\cdot\frac{3}{4}$. Solving, $p(B) = \frac{4}{9}$.

15. A box contains 2 red balls, 3 black balls, and 4 white balls. One is removed, and then another is removed. What is the probability that no black balls were drawn?

Let A be the event of no black ball on the first draw. Let B be the event of no black ball drawn on the second draw. Then $p(A) = \frac{6}{9}$, because 6 of the 9 balls are not black. If A has occurred, then 5 of the remaining 8 balls are not black, so $p(B|A) = \frac{5}{8}$. The probability that no black ball was drawn is then $p(A\cap B) = p(A)\cdot p(B|A) = \frac{6}{9}\frac{5}{8} = \frac{5}{12} = 41.\overline{6}\%$.

17. A coin is flipped 5 times, and there are more tails than heads. What is the probability that the first flip was a tail?

Solution: The sample space S is the set of length-5 lists made from symbols H and T, so $|S| = 2^5 = 32$. Let A be the event of there being more tails than heads, and let B be the event of the first flip being a tail. Thus the answer to the question will be $p(B|A)$. Fact 7.3 says $p(B|A) = \frac{p(A\cap B)}{p(A)}$, so we need to calculate $p(A \cap B)$ and $p(A)$. Note that $A \cap B$ is the event of more tails than heads **and** the first flip is a tail. If the first flip is a tail, and there are to be more tails than heads, then 2, 3 or 4 of the remaining 4 flips must be tails. The number of ways for this to happen is $\binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 6 + 4 + 1 = 11$, so $|A \cap B| = 11$. Considering $|A|$, to have more tails than heads, 3, 4 or 5 of the flips must be tails, and it follows that $|A| = \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 10 + 5 + 1 = 16$. To get our final answer, we have

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{|A\cap B|}{|S|}}{\frac{|A|}{|S|}} = \frac{|A \cap B|}{|A|} = \frac{11}{16} = 68.75\%.$$

19. A 5-card hand is dealt from a shuffled 52-card deck. Exactly 2 of the cards in the hand are hearts. Find the probability that all the cards in the hand are red.

Solution: Let A be the event of getting exactly 2 hearts in the hand. Let B be the event of all cards in the hand being red. Thus the answer to the question will be $p(B|A)$. Fact 7.3 says $p(B|A) = \frac{p(A\cap B)}{p(A)}$, so we need to calculate $p(A \cap B)$ and $p(A)$. Note that $A\cap B$ is the event of getting a 5-card hand that has 2 hearts and 3 diamonds. Thus $|A \cap B| = \binom{13}{2}\binom{13}{3}$. Also $|A| = \binom{13}{2}\binom{39}{3}$ (choose 2 hearts and 3 non-hearts). To get our final answer, we have

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{\frac{|A\cap B|}{|S|}}{\frac{|A|}{|S|}} = \frac{|A \cap B|}{|A|} = \frac{\binom{13}{2}\binom{13}{3}}{\binom{13}{2}\binom{39}{3}} = \frac{\binom{13}{3}}{\binom{39}{3}} = \frac{13 \cdot 12 \cdot 11}{39 \cdot 38 \cdot 37} \approx 3.13\%.$$

Section 7.4

1. A shuffled standard deck of cards is missing the ace of heats. You draw one card from the deck and observe its suit. Find the probability distribution.

Solution: $p(\heartsuit) = 12/51$, $p(\clubsuit) = 13/51$, $p(\diamondsuit) = 13/51$, $p(\spadesuit) = 13/51$.

3. A weighted coin is 1.5 times as likely to land on heads than tails. Find the probability distribution for the experiment of tossing the coin once. If you toss it three times, what are the chances of getting all heads? All tails? What are the chances of getting more heads than tails?

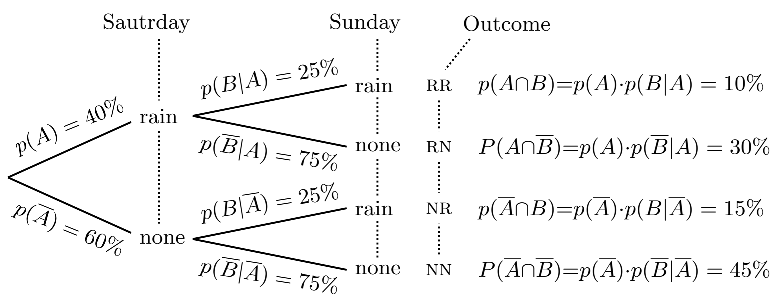
Solution: The sample space is $S = \{H, T\}$. Let $p(T) = x$, so $p(H) = 1.5x$. Because $p(T) + p(H) = 1$, we get $x + 1.5x = 1$, and solving for x yields $x = 2/5$.

Probability distribution: $p(T) = 2/5$, $\quad p(H) = 1.5p(T) = 3/2 \cdot 2/5 = 6/10 = 3/5$.

$p(HHH) = 3/5 \cdot 3/5 \cdot 3/5 = 27/125 = 21.6\%$.
$p(TTT) = 2/5 \cdot 2/5 \cdot 2/5 = 8/125 = 6.4\%$.
The chance of more heads than tails is $p(HHH)+p(HHT)+p(HTH)+p(THH) = 3/5 \cdot 3/5 \cdot 3/5 + 3(3/5 \cdot 3/5 \cdot 2/5) = 27/125 + 54/125 = 81/125 = 64.8\%$.

5. There is a 40% chance of rain on Saturday and a 25% chance of rain on Sunday. What is the probability that it will rain on at least one day of the weekend? (You may assume that the events “Rain on Saturday” and “Rain on Sunday” are independent events.)

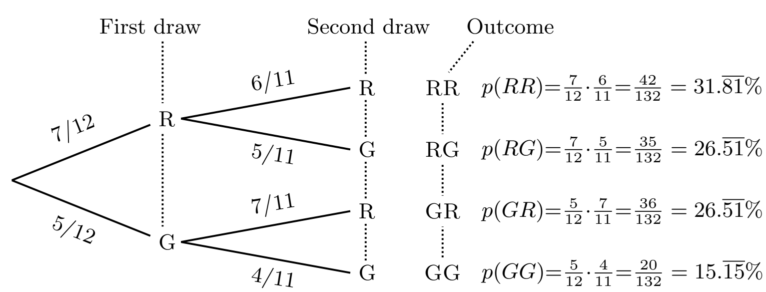
Solution: Say A is the event of rain on Saturday and B is the event of rain on Sunday. Then our sample space is $S = \{\text{RR, RN, NR, NN}\}$, and $A = \{\text{RR, RN}\}$ and $B = \{\text{RR, NR}\}$. Here is a probability tree for this.



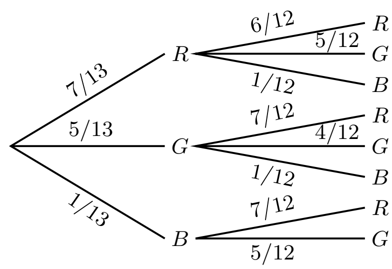
From this, the probability of rain over the weekend is $10\% + 30\% + 15\% = 55\%$.

If you got the answer without drawing a probability tree, then that is good. Another solution would be to calculate the probability of no rain over the weekend, which is $p(\overline{A}) \cdot p(\overline{B}|\overline{A}) = p(\overline{A}) \cdot p(\overline{B}) = 0.6 \cdot 0.75 = 45\%$. Then the probability of rain over the weekend is $1 - 0.45 = 55\%$.

7. A box contains 7 red balls and 5 green balls. You reach in and remove two balls, one after the other. Complete the probability tree.



9. A box contains 7 red balls, 5 green balls and 1 blue ball. You reach in and remove two balls, one after the other. Complete the probability tree.



$p(RR) = \frac{7}{13} \cdot \frac{6}{12} = \frac{42}{156} \approx 26.92\%$
$p(RG) = \frac{7}{13} \cdot \frac{5}{12} = \frac{35}{156} \approx 22.43\%$
$p(RB) = \frac{7}{13} \cdot \frac{1}{12} = \frac{7}{156} \approx 4.48\%$
$p(GR) = \frac{5}{13} \cdot \frac{7}{12} = \frac{35}{156} \approx 22.43\%$
$p(GG) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156} \approx 12.82\%$
$p(GB) = \frac{5}{13} \cdot \frac{1}{12} = \frac{5}{156} \approx 3.30\%$
$p(BR) = \frac{1}{13} \cdot \frac{7}{12} = \frac{7}{156} \approx 4.48\%$
$p(BG) = \frac{1}{13} \cdot \frac{5}{12} = \frac{5}{156} \approx 3.3\%$

11. A club consists of 60 men and 40 women. To fairly choose a president and a secretary, names of all members are put into a hat and two names are drawn. The first name drawn is the president, and the second name drawn is the secretary. What is the probability that the president and the secretary have the same gender?

Solution: Say the sample space is $S = \{\text{MM, MW, WM, WW}\}$ where the first letter indicates the gender of the first draw, and the second letter indicates the gender of the second draw. Then $p(\text{MM}) = \frac{60}{100} \cdot \frac{59}{99} = \frac{3540}{9900}$ and $p(\text{WW}) = \frac{40}{100} \cdot \frac{39}{99} = \frac{1560}{9900}$. Thus the probability of both offices being the same gender is $p(\{\text{MM, WW}\}) = \frac{3540}{9900} + \frac{1560}{9900} = \frac{5100}{9900} = 51.\overline{51}\%$.

13. At a certain college, 30% of the students are freshmen. Also, 80% of the freshmen live on campus, while only 60% of the non-freshman students live on campus. A student is chosen at random. What is the probability that the student is a freshman who lives off campus?

Solution: Let A be the event of choosing a freshman. Let B be the event of choosing someone who lives on campus. The given information states that $p(A) = 30\%$ and $p(\overline{B}|A) = 20\%$. (If you chose a freshman, there is an 80% chance he or she lives on campus, so there is a 20% chance he or she lives off campus.)

The problem is asking for $p(A \cap \overline{B})$. Now, $p(A \cap \overline{B}) = p(A) \cdot p(\overline{B}|A) = 0.30 \cdot 0.20 = 6\%$. Thus there is a 6% chance of choosing a freshman who lives off campus.

15. An unfair coin is twice as likely to land on heads as tails.
If you toss it 10 times what are the chances that you will get exactly 6 heads?
If you toss it n times what are the chances that you will get exactly k heads?

Solution: The sample space for tossing once is $S = \{H, T\}$, with probability distribution $p(H) = 2/3$ and $p(T) = 1/3$. The chance of getting the outcome HHHHHHTTTT (with 6 heads) is $(2/3)^6(1/3)^4 = 2^6/3^{10}$. Likewise, the chance of getting TTHHHHHHTT is $(1/3)^2(2/3)^6(1/3)^2 = 2^6/3^{10}$, etc. Since there are $\binom{10}{6}$ ways to get exactly 6 heads in 10 tosses, the probability of getting exactly 6 heads is $\binom{10}{6}\frac{2^6}{3^{10}} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1}\frac{2^6}{3^{10}} \approx 22.76\%$.

To answer the second question we reason as above to get a probability of $\binom{n}{k}\frac{2^k}{3^{10}}$.

Section 7.5

1. At a certain college, 40% of the students are male, and 60% are female. Also, 20% of the males are transfer students, and 10% of the females are transfer students. A student is chosen at random. If the student is a transfer student, what is the probability that the student is female?

Solution: Let S be the set of all students, S_1 be the set of female students, and S_2 be the set of male students. Then $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Let $E \subseteq S$ be the set of transfer students. The problem asks for $p(S_1|E)$. Bayes' theorem applies and we get

$$p(S_1|E) = \frac{p(S_1) \cdot p(E|S_1)}{p(S_1) \cdot p(E|S_1) + p(S_2) \cdot p(E|S_2)} = \frac{0.60 \cdot 0.10}{0.60 \cdot 0.10 + 0.40 \cdot 0.20} = \frac{0.6}{0.8} = 75\%.$$

3. A jar contains 4 red balls and 5 white balls. A random ball is removed, and then another is removed. If the second ball was red, what is the probability that the first ball was red?

Solution: The sample space is $S = \{\text{RR}, \text{RW}, \text{WR}, \text{WW}\}$, where the first letter is the color of the first ball and the second letter is the color of the second ball. Let $S_1 = \{\text{RR}, \text{RW}\}$ be the event of the first ball being red. Let $S_2 = \{\text{WR}, \text{WW}\}$ be the event of the first ball being white. Let $E = \{\text{RR}, \text{WR}\}$ be the event of the second ball being red. The answer to the question is thus $p(S_1 \mid E)$. As $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$, Bayes' formula applies, and it gives

$$\begin{aligned} p(S_1 \mid E) &= \frac{p(S_1) \cdot p(E \mid S_1)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)} \\ &= \frac{\frac{4}{9} \cdot \frac{3}{8}}{\frac{4}{9} \cdot \frac{3}{8} + \frac{5}{9} \cdot \frac{4}{8}} = \frac{\frac{1}{6}}{\frac{3}{18} + \frac{5}{18}} = \frac{\frac{1}{6}}{\frac{4}{9}} = \frac{3}{8} = 37.5\%. \end{aligned}$$

So there's a 37.5% chance that the first ball was red if the second is red.

5. A retail store obtains battery packs from two distributers, Distributer A and Distributer B. On a particular day, 70% of the packs in the store are from Distributer A, and 30% are from Distributer B. It happens that 1% of the packs from Distributer A contain a defective battery, and 2% of the packs from Distributer B contain a defective battery. A consumer buys a pack that contains a defective battery. What is the probability that it came from Distributer B?

Let S be the set of all battery packs in the store at the moment the consumer picked one. Then $S = S_1 \cup S_2$, where S_1 is the set of packs from Distributer A, and S_2 is the set of packs from Distributer B. Then $p(S_1) = 70\%$ and $p(S_2) = 30\%$. Let $E \subset S$ be the set of packs that contain a defective battery. (So E is the event of the consumer picking a defective pack.) The problem states that $p(E|S_1) = 1\%$ and $p(E|S_2) = 2\%$. The answer to the question is

$$p(S_2 \mid E) = \frac{p(S_2) \cdot p(E \mid S_2)}{p(S_1) \cdot p(E \mid S_1) + p(S_2) \cdot p(E \mid S_2)} = \frac{0.3 \cdot 0.02}{0.7 \cdot 0.01 + 0.3 \cdot 0.02} = 46.15\%$$

7. Solution: $p(A|B) = \dfrac{p(A \cap B)}{p(B)} = \dfrac{p(A \cap B)}{p(A)} \dfrac{p(A)}{p(B)} = \dfrac{p(B|A) \cdot p(A)}{p(B)}.$