Chapter 20

Review of Real-Valued Functions

In Chapter 8 we saw how the efficiency of algorithms is measured by functions of the size of their inputs. For example, the sequential search algorithm (page 224) needs $f(n) = 2 + 2n$ steps (in the worst case) to search a list of length n. Binary search (page 226) needs only $g(n) = 3 + 2\log_2(n)$ steps.

Using functions of input size to study algorithms performance is important in computer science, and is a key ingredient of the final chapters of this book. This chapter's purpose is to review the basic properties of functions that arise most naturally in this context, namely polynomial, exponential and logarithm functions. There is nothing really new here; you have studied such functions for years. But if you are a bit rusty with, say, exponents and logarithms, then this chapter is a refresher. You can skip it if you already have a good grip on the algebra of functions.

Chapter 18 developed a theory of functions $f : A \to B$ from one set to another. Here we deal with functions $f : \mathbb{R} \to \mathbb{R}$, where A and B are the set $\mathbb{R}$, or perhaps intervals on the real line, as in $f : \mathbb{R} \to [0, \infty)$. Actually, for the uses to which we will later apply such functions, the domain is input size (a positive integer) and the co-domain is is the number of steps executed by an algorithm (also a positive integer). In this context, they are properly viewed as functions $f : \mathbb{N} \to \mathbb{N}$. However, because $\mathbb{N} \subseteq \mathbb{R}$, we typically view the domain co-domain as sets of real numbers.

Topics from Chapter 18 that play a role in this chapter include domain, range, co-domain, as well as injective, surjective, bijective and inverse functions.

Let us begin with exponents.

20.1 Exponent Review

We start at the beginning. In an expression like a^n, where a is raised to the nth power, a is called the **base** and n is the **exponent.** If n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}.$$

This would be too elementary to mention except that every exponent property flows from it. For example,

$$(ab)^n = \underbrace{(ab)\cdot(ab)\cdot(ab)\cdots(ab)}_{n \text{ times}} = \underbrace{a\cdot a\cdot a\cdots a}_{n \text{ times}}\cdot\underbrace{b\cdot b\cdot b\cdots b}_{n \text{ times}} = a^n b^n.$$

Therefore $\boxed{(ab)^n = a^n b^n}$. Also, $\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdots\frac{a}{b} = \frac{a^n}{b^n}$, so $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$. And $\boxed{a^m a^n = a^{m+n}}$ because $a^m a^n = \underbrace{a\cdot a\cdot a\cdots a}_{m \text{ times}}\cdot\underbrace{a\cdot a\cdot a\cdots a}_{n \text{ times}} = a^{m+n}$.

Assuming for the moment that $m > n$, we have

$$\frac{a^m}{a^n} = \frac{\overbrace{a\cdot a\cdot a\cdot a\cdot a\cdots a}^{m \text{ times}}}{\underbrace{a\cdot a\cdot a\cdots a}_{n \text{ times}}} = \underbrace{a\cdot a\cdot\cdots a}_{m-n \text{ times}} = a^{m-n}$$

because the a's on the bottom cancel with a's on top, leaving $m-n$ a's on top. Also notice that $\boxed{(a^n)^m = a^{nm}}$ because

$$(a^n)^m = \overbrace{(\underbrace{a\cdot a\cdot a\cdots a}_{n \text{ times}})\cdot(\underbrace{a\cdot a\cdot a\cdots a}_{n \text{ times}})\cdots(\underbrace{a\cdot a\cdot a\cdots a}_{n \text{ times}})}^{m \text{ groups of } n\ a\text{'s}} = a^{nm}.$$

We have just verified the following fundamental Laws of exponents.

Fact 20.1. Basic Laws of Exponents

$$a^1 = a \qquad (ab)^n = a^n b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^n)^m = a^{mn}$$

So far we have assumed n is a positive integer because in $a^n = a\cdot a\cdots a$ we cannot multiply a times itself a negative or fractional number of times. Still, a^n makes sense for n zero, negative or fractional. Trusting the above property $a^{m-n} = \frac{a^m}{a^n}$ yields

$$a^0 = a^{1-1} = \frac{a^1}{a^1} = 1, \quad (\text{provided } a \neq 0)$$
$$a^{-n} = a^{0-n} = \frac{a^0}{a^n} = \frac{1}{a^n}.$$

Notice 0^0 is undefined because $0^0 = 0^{1-1} = \frac{0^1}{0^1} = \frac{0}{0}$, which is undefined. But we can find a^n when n is 0 (and $a \neq 0$) or negative, as in $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$. In essence we just multiplied 2 times itself -3 times! Also note $a^{-1} = \frac{1}{a}$.

What about fractional powers, like $a^{m/n}$ or $a^{1/n}$? Trusting $(a^n)^m = a^{nm}$ yields

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n}\cdot n} = a^1 = a.$$

In words, $a^{\frac{1}{n}}$ is a number that, if raised to the power of n, results in a. This means $a^{1/n} = \sqrt[n]{a}$. For example, $16^{1/4} = \sqrt[4]{16} = 2$ and $2^{1/2} = \sqrt{2}$. Further, $a^{\frac{m}{n}} = a^{\frac{1}{n}m} = \left(a^{\frac{1}{n}}\right)^m = \sqrt[n]{a}^{\,m} = \sqrt[n]{a^m}$. Let's summarize all of this.

Fact 20.2. Laws of zero, negative and rational exponents

$$a^0 = 1 \quad (\text{if } a \neq 0) \qquad a^{-n} = \frac{1}{a^n} \qquad a^{-1} = \frac{1}{a}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = \sqrt[n]{a}^{\,m} = \sqrt[n]{a^m}$$

The boxed equations hold for any rational m and n, positive or negative.

Example 20.1. Knowing the above laws of exponents means we can evaluate many expressions without a calculator. Suppose we are confronted with $16^{-1.5}$. What number is this? We reckon as follows

$$16^{-1.5} = 16^{-3/2} = \frac{1}{16^{3/2}} = \frac{1}{\sqrt{16}^3} = \frac{1}{4^3} = \frac{1}{64}.$$

For another example, $8^{-1.5} = 8^{-3/2} = \frac{1}{8^{3/2}} = \frac{1}{\sqrt{8}^3} = \frac{1}{(2\sqrt{2})^3} = \frac{1}{2^3\sqrt{2}^3} = \frac{1}{16\sqrt{2}}$.

Also, $(-8)^{5/3} = \sqrt[3]{-8}^{\,5} = (-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$. ✍

Exercises for Section 20.1

Work the following exponents with pencil and paper alone. Then compare your answer to a calculator's to verify that the calculator is working properly.

1. $25^{1/2}$ **2.** $4^{1/2}$ **3.** $\frac{1}{4}^{1/2}$ **4.** $27^{1/3}$
5. $(-27)^{1/3}$ **6.** $(27)^{-1/3}$ **7.** $(-27)^{4/3}$ **8.** 2^{-1}
9. 2^{-2} **10.** 2^{-3} **11.** $\frac{1}{2}^{-1}$ **12.** $\frac{1}{2}^{-2}$
13. $\frac{1}{2}^{-3}$ **14.** $\frac{1}{4}^{-1/2}$ **15.** $\sqrt{2}^6$ **16.** $\left(\left(\frac{2}{3}\right)^{\frac{3}{2}}\right)^2$
17. $\left(\frac{3^9}{3^7}^{\frac{2}{3}}\right)^3$ **18.** $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ **19.** $\left(\frac{4}{9}\right)^{-1/2}$ **20.** $\left(\frac{\sqrt{3}}{2}\right)^{-4}$
21. $\frac{\sqrt{3}^{100}}{\sqrt{3}^{94}}$

20.2 Linear Functions, Power Functions and Polynomials

Some functions (and families of functions) are so elemental that they become part of our daily mathematical vocabulary. Here is a quick inventory.

We begin with linear functions. A **linear function** is a function $f : \mathbb{R} \to \mathbb{R}$ having the form $f(x) = mx + b$, where m and b are constants. The graph of this function is a straight line with slope m and y-intercept b. See Figure 20.1.

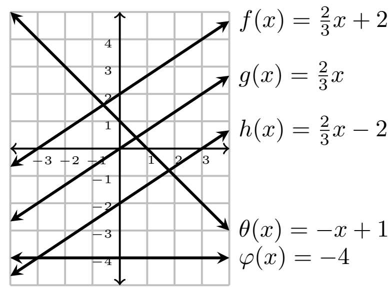


Fig. 20.1 Some linear functions.

In $f(x) = mx+b$ it is of course possible that $m = 0$, giving the function $f(x) = b$. This is called a **constant function**; no matter what the input x is, the output is always the same number b. The graph of this function is a horizontal line (slope 0) passing through the point b on the y-axis. The constant function $\varphi(x) = -4$ is illustrated above. You could write is as $\varphi(x) = 0 \cdot x - 4$ and regard it as the rule *multiply x by zero and subtract* 4.

A **power function** is a function of form $f(x) = x^n$, where the exponent n is a constant. Figure 20.2 shows a few examples for $n \in \mathbb{N}$. It is important to *internalize* (not just memorize) these graphs. Take time to understand why the graphs look the way they do. Notice that when n is even x^n is positive for any x, so the graph lies above the x axis in those cases. By contrast, for odd n the value x^n is negative whenever x is negative; thus a portion of these graphs is below the x-axis.

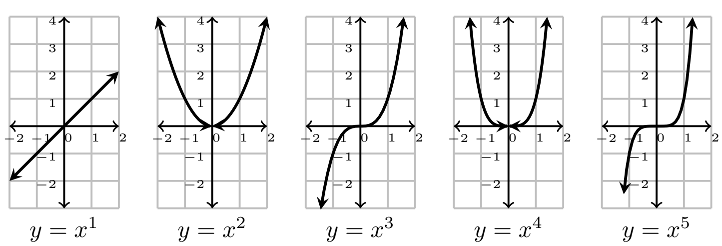


Fig. 20.2 Power functions $f(x) = x^n$. In each case the domain is all real numbers, $\mathbb{R}$. If n is even the range is $[0, \infty)$. If n is odd the range is $\mathbb{R}$.

It is of course possible to have power functions $f(x) = x^n$ for which n is not an integer. For example, $f(x) = x^{1/2} = \sqrt{x}$, and in general $f(x) = x^{a/b} = \sqrt[b]{x}^{\,a}$. Notice that if a and b are positive integers, then $f(x) = x^{a/b} = \sqrt[b]{x}^{\,a}$ grows arbitrarily large as x increases: For any positive y (no matter how large), $x > \sqrt[a]{y}^{\,b}$ implies $f(x) > \sqrt[b]{\sqrt[a]{y}^{\,b}}^{\,a} = y$. This illustrates an important fact.

Fact 20.3. For powers $n > 0$, a power function $f(x) = x^n$ increases to ∞ as x increases to ∞. In other words, if $n > 0$, then $\lim_{x\to\infty} x^n = \infty$.

A **polynomial function** is a sum of multiples of integer powers of x, plus a constant term (which could be 0). So $f(x) = x^4 - 2x^2 + \pi x + 2$ is a polynomial with constant term 2, and $g(x) = 5x^2 + 3x - 1$ is a polynomial with constant term -1. The **degree** of a polynomial is its highest power of x, so $f(x)$ has degree 4 and $g(x)$ has degree 2. A linear function, such as $h(x) = 3x + 7$, is a polynomial of degree 1, as $h(x) = 3x^1 + 7$. We regard a constant function $f(x) = b$ as a polynomial of degree 0, as $b = bx^0$ (for $x \neq 0$).

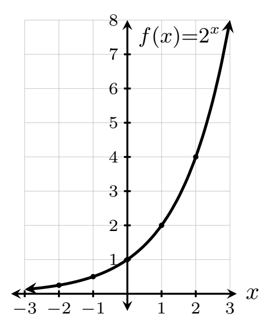
20.3 Exponential Functions

Interchanging the x and the 2 in the power function $f(x) = x^2$ gives a new function $f(x) = 2^x$. A function like this one, a constant raised to a variable power, is called an *exponential function*.

An **exponential function** is one of form $f(x) = a^x$, where a is a positive constant, called the **base** of the exponential function. For example, $f(x) = 2^x$ and $f(x) = 3^x$ are exponential functions, as is $f(x) = \frac{1}{2}^x$. If we let $a = 1$ in $f(x) = a^x$ we get $f(x) = 1^x = 1$, which is, in fact, a linear function. For this reason we agree that the base of an exponential function is never 1.

Let's graph the exponential function $f(x) = 2^x$. Below is a table with some sample x and $f(x)$ values. The resulting graph is on the right.

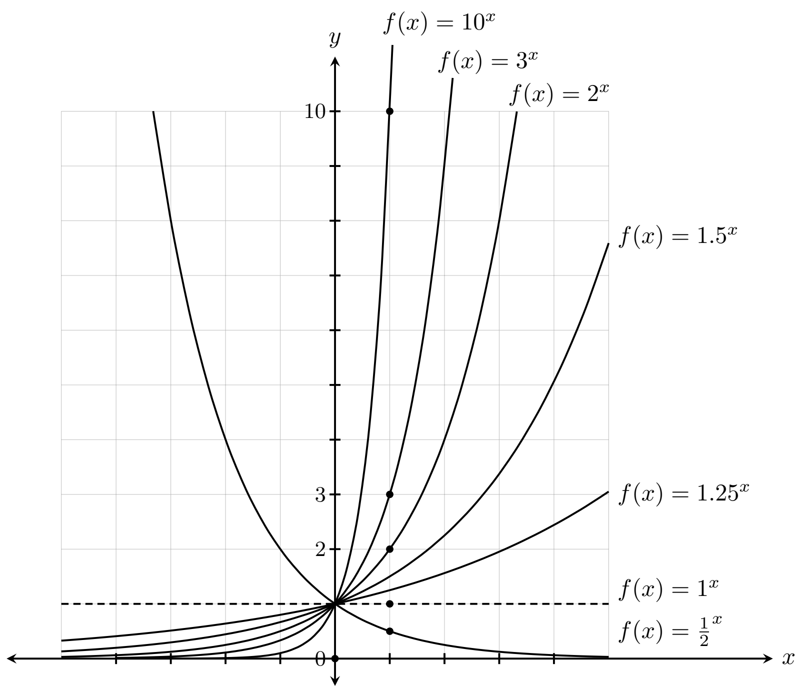
x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	1/8	1/4	1/2	1	2	4	8



Notice that $f(x) = 2^x$ is positive for any x, but gets closer to zero the as x moves in the negative direction. But $2^x > 0$ for any x, so the graph never touches the x-axis.

Working with exponential functions requires fluency with the exponent properties of Section 20.1. For example, if $f(x) = 2^x$, then $f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ and $f\left(\frac{3}{2}\right) = 2^{\frac{3}{2}} = \sqrt{2}^3 = \sqrt{2}\sqrt{2}\sqrt{2} = 2\sqrt{2}$.

Several exponential functions are graphed below. These graphs underscore the fact that the domain of any exponential function is $\mathbb{R}$. The range is $(0, \infty)$. The y-intercept of any exponential function is 1.



Notice that if the base of an exponential function is less than 1, like in $f(x) = \frac{1}{2}^x$, then the graph *decreases* as x increases. If in doubt, write a table for this function and graph it. (This involves using the formula $a^{-x} = \frac{1}{a^x}$.) But if the base a is greater than 1, then $f(x) = a^x$ grows very quickly as x increases.

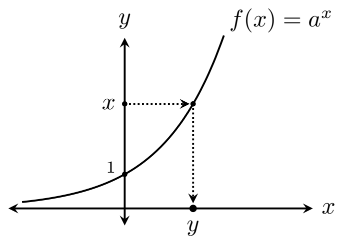
Fact 20.4. If $a > 1$, the exponential function $f(x) = a^x$ increases to ∞ as x increases to ∞.

Next we investigate inverses of exponential functions. They are called *logarithms*.

20.4 Logarithmic Functions

Now we apply the ideas of Chapter 18 to explore inverses of exponential functions. Such inverses are called *logarithmic functions*, or just *logarithms*. An exponential function $f(x) = a^x$, viewed as $f : \mathbb{R} \to (0, \infty)$, is bijective and thus has an inverse.

As illustrated below, this inverse sends any number x to the number y for which $f(y) = x$, that is, for which $a^y = x$.



In other words, the rule is $f^{-1}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } a^y = x \end{pmatrix}$.

From this it seems that a better name for f^{-1} might be $a^{\square}$, for then

$$a^{\square}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } a^y = x \end{pmatrix}.$$

The idea is that $a^{\square}(x)$ is the number y that goes in the box so that $a^y = x$. Using $a^{\square}$ as the name of f^{-1} thus puts the meaning of f^{-1} into its name. We therefore will use the symbol $a^{\square}$ instead of f^{-1} for the inverse of $f(x) = a^x$.

For example, the inverse of $f(x) = 2^x$ is a function called $2^{\square}$, where

$$2^{\square}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } 2^y = x \end{pmatrix}.$$

Consider $2^{\square}(8)$. Putting 3 in the box gives $2^3 = 8$, so $2^{\square}(8) = 3$. Similarly

$$\begin{aligned} 2^{\square}(16) &= 4 \quad \text{because } 2^4 = 16, \\ 2^{\square}(4) &= 2 \quad \text{because } 2^2 = 4, \\ 2^{\square}(2) &= 1 \quad \text{because } 2^1 = 2, \\ 2^{\square}(0.5) &= -1 \quad \text{because } 2^{-1} = \tfrac{1}{2} = 0.5. \end{aligned}$$

In the same spirit the inverse of $f(x) = 10^x$ is a function called $10^{\square}$, and

$$10^{\square}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } 10^y = x \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} 10^{\square}(1000) &= 3 \quad \text{because } 10^3 = 1000, \\ 10^{\square}(10) &= 1 \quad \text{because } 10^1 = 10, \\ 10^{\square}(0.1) &= -1 \quad \text{because } 10^{-1} = \tfrac{1}{10} = 0.1. \end{aligned}$$

Given a power 10^p of 10 we have $10^{\square}\left(10^p\right) = p$. For example,

$$\begin{aligned} 10^{\square}\,(100) &= 10^{\square}\left(10^2\right) = 2, \\ 10^{\square}\left(\sqrt{10}\right) &= 10^{\square}\left(10^{1/2}\right) = \tfrac{1}{2}. \end{aligned}$$

But doing, say, $10^{\square}(15)$ is not so easy because 15 is not an obvious power of 10. We will revisit this at the end of the section.

In general, the inverse of $f(x) = a^x$ is a function called $a^{\square}$, pronounced "*a box*," and defined as

$$a^{\square}(x) = \begin{pmatrix} \text{the number } y \\ \text{for which } a^y = x \end{pmatrix}.$$

You can always compute $a^{\square}$ of a power of a in your head because $a^{\square}(a^p) = p$.

The notation $a^{\square}$ is nice because it reminds us of the meaning of the function. But this book is probably the only place that you will ever see the symbol $a^{\square}$. Every other textbook—in fact all of the civilized world—uses the symbol $\log_a$ instead of $a^{\square}$, and calls it the *logarithm to base a.*

Definition 20.1. For $a > 0$ and $a \neq 1$, the **logarithm to base a** is the function

$$\log_a(x) = a^{\square}(x) = \begin{pmatrix} \text{number } y \text{ for} \\ \text{which } a^y = x \end{pmatrix}.$$

The function $\log_a$ is pronounced "*log base a.*" It is the inverse of $f(x) = a^x$.

Here are some examples.

$$\begin{array}{ll} \log_2(8) = 2^{\square}(8) = 3 & \log_5(125) = 5^{\square}(125) = 3 \\ \log_2(4) = 2^{\square}(4) = 2 & \log_5(25) = 5^{\square}(25) = 2 \\ \log_2(2) = 2^{\square}(2) = 1 & \log_5(5) = 5^{\square}(5) = 1 \\ \log_2(1) = 2^{\square}(1) = 0 & \log_5(1) = 5^{\square}(1) = 0 \end{array}$$

To repeat, $\log_a$ and $a^{\square}$ are different names for the same function. We will bow to convention and use $\log_a$, mostly. But we will revert to $a^{\square}$ whenever it makes the discussion clearer.

Understanding the graphs of logarithm functions is important. Recall from algebra that the graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected across the line $y = x$. Because $\log_a$ is the inverse of $f(x) = a^x$, its graph is the graph of $y = a^x$ reflected across the line $y = x$, as illustrated in Figure 20.3.

Take note that the domain of $\log_a$ is all positive numbers $(0, \infty)$ because this is the *range* of a^x. Likewise the range of $\log_a$ is the domain of a^x, which is $\mathbb{R}$. Also, because $\log_a(1) = a^{\square}(1) = 0$, the x-intercept of $y = \log_a(x)$ is 1.

The logarithm function $\log_{10}$ to base 10 occurs frequently enough that it is abbreviated as log and called the *common logarithm.*

Definition 20.2. The **common logarithm**, denoted log, is the function

$$\log(x) = \log_{10}(x) = 10^{\square}(x).$$

Most calculators have a $\boxed{\log}$ button for the common logarithm. Test your calculator by confirming $\log(1000) = 3$ and $\log(0.1) = -1$. The button will also tell

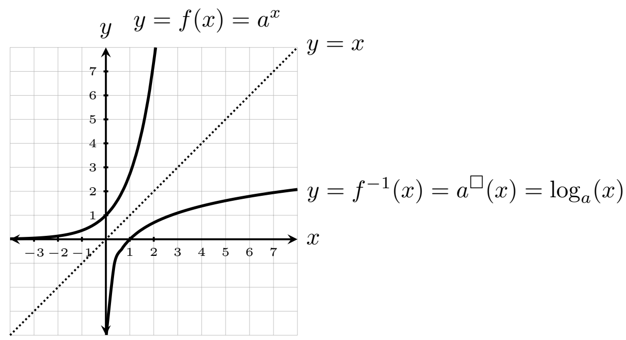


Fig. 20.3 The exponential function $y = a^x$ and its inverse $y = \log_a(x)$.

you that $\log(15) \approx 1.17609125$. In other words $10^{\square}(15) \approx 1.17609125$, which means $10^{1.17609125} \approx 15$, a fact with which your calculator will concur.

One comment. Convention allows for $\log_a x$ in the place of $\log_a(x)$, that is, the parentheses may be dropped. We will tend to use them.

Logarithms have many important properties, which we now review. To start, for any x it is obvious that $a^{\square}(a^x) = x$ because x is what must go into the box so that a to that power equals a^x. So we have

$$\begin{aligned} a^{\square}\left(a^x\right) &= x, \\ \log_a\left(a^x\right) &= x. \end{aligned} \tag{20.1}$$

This simply reflects the fact that $f^{-1}\big(f(x)\big) = x$ for the function $f(x) = a^x$.

Next consider the expression $a^{a^{\square}(x)}$. Here a is being raised to the power $a^{\square}(x)$, which is literally the power a must be raised to to give x. Therefore

$$\begin{aligned} a^{a^{\square}(x)} &= x, \\ a^{\log_b(x)} &= x \end{aligned} \tag{20.2}$$

for any x in the domain of $a^{\square}$. This is just saying $f\big(f^{-1}(x)\big) = x$.

The x in Equations (20.1) and (20.2) can be any appropriate quantity or expression. It is reasonable to think of these equations as saying

$$a^{\square}\left(a^{\blacksquare}\right) = \blacksquare \qquad \text{and} \qquad a^{a^{\square}(\blacksquare)} = \blacksquare,$$

where the gray rectangle can represent an arbitrary expression. Thus $a^{\square}\left(a^{x+y^2+3}\right) = x + y^2 + 3$ and $a^{a^{\square}\left(5x+1\right)} = 5x + 1$.

Next we verify a fundamental formula for $\log_a(xy)$, that is, $a^{\square}(xy)$. Notice

$$\begin{aligned} a^{\square}(xy) &= a^{\square}\left(a^{a^{\square}(x)}\, a^{a^{\square}(y)}\right) && \text{...... because } x = a^{a^{\square}(x)} \text{ and } y = a^{a^{\square}(y)} \\ &= a^{\square}\left(a^{a^{\square}(x)+a^{\square}(y)}\right) && \text{..................... because } a^c a^d = a^{c+d} \\ &= a^{\square}(x) + a^{\square}(y) && \text{................... using } a^{\square}\left(a^{\blacksquare}\right) = \blacksquare \end{aligned}$$

We have therefore established

$$\begin{aligned} a^{\square}(xy) &= a^{\square}(x) + a^{\square}(y), \\ \log_a(xy) &= \log_a(x) + \log_a(y). \end{aligned} \tag{20.3}$$

By the same reasoning you can also show $a^{\square}\left(\frac{x}{y}\right) = a^{\square}(x) - a^{\square}(y)$, that is,

$$\begin{aligned} a^{\square}\left(\frac{x}{y}\right) &= a^{\square}(x) - a^{\square}(y), \\ \log_a\left(\frac{x}{y}\right) &= \log_a(x) - \log_a(y). \end{aligned} \tag{20.4}$$

Applying $a^{\square}(1) = 0$ to this yields $a^{\square}\left(\frac{1}{y}\right) = a^{\square}(1) - a^{\square}(y) = -a^{\square}(y)$, so

$$\begin{aligned} a^{\square}\left(\frac{1}{y}\right) &= -a^{\square}(y), \\ \log_a\left(\frac{1}{y}\right) &= -\log_a(y). \end{aligned} \tag{20.5}$$

Here is a summary of what we have established so far.

Fact 20.5. Logarithm Laws

$$\begin{array}{ll} \log_a(a^x) = x & \log_a(1) = 0 \\ a^{\log_a(x)} = x & \log_a(a) = 1 \\ \log_a(xy) = \log_a(x) + \log_a(y) & \log_a(x^y) = y\log_a(x) \\ \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) & \log_a\left(\frac{1}{y}\right) = -\log_a(y) \end{array}$$

The one law in this list that we have not yet verified is $\log_a(x^y) = y\log_a(x)$. This rule says that taking $\log_a$ of x^y converts the *exponent* y to a *product*. Because products tend to be simpler than exponents, this property can be tremendously useful. To verify it, just notice that

$$\begin{aligned} a^{\square}(x^y) &= a^{\square}\left(\left(a^{a^{\square}(x)}\right)^y\right) && \text{.................. because } x = a^{a^{\square}(x)} \\ &= a^{\square}\left(a^{y\,a^{\square}(x)}\right) && \text{................ because } \left(a^b\right)^y = a^{yb} \\ &= y\,a^{\square}(x) && \text{.............. using } a^{\square}\left(a^{\blacksquare}\right) = \blacksquare \end{aligned}$$

Therefore $a^{\square}(x^y) = y\,a^{\square}(x)$, or $\log_a(x^y) = y\,\log_a(x)$, as listed above.

By the above laws, certain expressions involving logarithms can be transformed into simpler expressions. Some examples follow.

Example 20.2. Simplify $\log_2(28x) - \log_2(7x)$.

To solve this we use the laws of logarithms to get

$$\begin{aligned}\log_2(28x) - \log_2(7x) &= \log_2\left(\frac{28x}{7x}\right)\\ &= \log_2(4)\\ &= 2.\end{aligned}$$

✍

As mentioned above, the law $\log_a(x^r) = r\log_a(x)$ is extremely useful because it means taking $\log_a$ of x^r converts the exponent r to a product. Consequently $\log_a$ can be used to solve an equation for a quantity that appears as an exponent, as in the next example.

Example 20.3. Solve the equation $5^{x+7} = 2^x$. In other words we want to find the value of x that makes this true. Since x occurs as an exponent, we take $\log_{10}$ of both sides and simplify with log laws.

$$\begin{aligned}\log\left(5^{x+7}\right) &= \log\left(2^x\right)\\ (x+7)\cdot\log(5) &= x\cdot\log(2)\\ x\log(5) + 7\log(5) &= x\log(2)\\ x\log(5) - x\log(2) &= -7\log(5)\\ x\big(\log(5) - \log(2)\big) &= -7\log(5)\\ x &= \frac{-7\log(5)}{\log(5) - \log(2)} \approx -12.2952955815\end{aligned}$$

where we have used a calculator in the final step. ✍

Example 20.4. Suppose a is positive. Solve the equation $a^y = x$ for y.

The variable y is an exponent, so we take log of both sides and simplify.

$$\begin{aligned}\log\left(a^y\right) &= \log(x)\\ y\log(a) &= \log(x)\\ y &= \frac{\log(x)}{\log(a)}\end{aligned}$$

Therefore, in terms of x and a, the quantity y is the number $\frac{\log(x)}{\log(a)}$. ✍

Example 20.4 would have been quicker if we had used $\log_a$ instead of log. Lets's do the same problem again with this alternative approach.

Example 20.5. Suppose a is positive. Solve the equation $a^y = x$ for y.

The variable y is an exponent, so we take $\log_a$ of both sides and simplify.

$$\begin{aligned}\log_a\left(a^y\right) &= \log_b(x)\\ y &= \log_a(x)\end{aligned}$$

Therefore, in terms of x and a, the quantity y is the number $\log_a(x)$. ✍

Examples 20.4 and 20.5 say the solution of $a^y = x$ can be expressed either as $\frac{\log(x)}{\log(a)}$ or $\log_a(x)$. The first solution may be preferable, as your calculator has no $\boxed{\log_a}$ button. But what is significant is that these two methods arrive at the *same* solution, which is to say $\log_a(x) = \frac{\log(x)}{\log(a)}$. To summarize:

Fact 20.6. Change of Base Formula

$$\log_a(x) = \frac{\log(x)}{\log(a)}$$

The change of base formula says that a logarithm $\log_a(x)$ to *any* base a can be expressed entirely in terms of $\log_{10}$.

Example 20.6. By the change of base formula, $\log_2(5) = \dfrac{\log(5)}{\log(2)} \approx \dfrac{1.698970}{0.301029} =$ 2.3219280. This seems about right because $\log_2(5) = 2^{\square}(5)$ is the number y for which $2^y = 5$. Now, $2^2 = 4 < 5 < 8 = 2^3$, so y should be between 2 and 3. This example shows in fact $y = 2.3219280$, to seven decimal places. ✍

Exercises for Section 20.4

Find the following logarithms with pencil and paper (no calculator).

1. $\log_3(81)$
2. $\log_3\left(\frac{1}{9}\right)$
3. $\log_3(\sqrt{3})$
4. $\log_3\left(\frac{1}{\sqrt{3}}\right)$
5. $\log_3(1)$
6. $\log(1000)$
7. $\log(\sqrt[3]{10})$
8. $\log(\sqrt[3]{100})$
9. $\log(0.01)$
10. $\log(1)$
11. $\log_4(4)$
12. $\log_4(2)$
13. $\log_4(\sqrt{2})$
14. $\log_4(16)$
15. $\log_4(8)$

Simplify the following expressions.

16. $\log_2(2^{\sin(x)})$
17. $10^{\log(5x+1)}$
18. $\log\left(10x^{10}\right)$
19. $\log(2) + \log(5)$
20. $\log(2) + \log(2x) + \log(25x)$
21. $\log_2(2) - \log_2(5x) + \log_2(20x)$

Write the following expressions as a single logarithm.

22. $5\log_2(x^3+1) + \log_2(x) - \log_2(3)$
23. $\log_2(\sin(x)) + \frac{1}{2}\log_2(4x) - 3\log_2(3)$
24. $2 + \log(5) + 2\log(7)$
25. $\log(2x) + \log(5x)$

Break up the following expressions into simpler logarithms.

26. $\log_2\left(x^3(x+1)\right)$
27. $\log_2\left((x+5)^4 x^7 \sqrt{x+1}\right)$
28. $\log\left(\sqrt{x}(x+3)^6\right)$
29. $\log_3\left(\frac{3}{5\sqrt[3]{x}}\right)$

Use the change of base formula to express the following logarithms in terms of log.

30. $\log_2(5)$
31. $\log_3(5)$
32. $\log_4(5)$
33. $\log_5(5)$
34. $\log(8)$
35. $\log_9(10)$
36. $\log(10)$
37. $\log_2(33)$
38. $\log_3(8)$
39. $\log_3(9)$

20.5 The Triangle Inequality

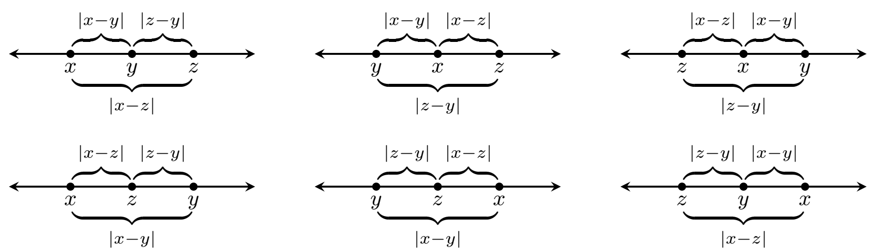
The next chapter uses absolute value extensively, so it is fitting to quickly review it. As you know, the absolute value of a real number x is the non-negative number

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0. \end{cases}$$

Fundamental properties of absolute value include $|xy| = |x|\cdot|y|$ and $x \leq |x|$. Another property—used often in proofs—is the *triangle inequality*:

Theorem 20.7. (Triangle inequality) *If $x, y, z \in \mathbb{R}$, then $|x-y| \leq |x-z|+|z-y|$.*

Proof. The name *triangle inequality* comes from the fact that the theorem can be interpreted as asserting that for any "triangle" on the number line, the length of any side never exceeds the sum of the lengths of the other two sides. Indeed, the distance between any two numbers $a, b \in \mathbb{R}$ is $|a-b|$. With this in mind, observe in the diagrams below that regardless of the order of x, y, z on the number line, the inequality $|x-y| \leq |x-z| + |z-y|$ holds.



(These diagrams show x, y, z as distinct points. If $x = y$, $x = z$ or $y = z$, then $|x-y| \leq |x-z| + |z-y|$ holds automatically.) □

The triangle inequality says the shortest route from x to y avoids z unless z lies between x and y. Several useful results flow from it. Put $z = 0$ to get

$$|x-y| \leq |x| + |y| \quad \text{for any } x, y \in \mathbb{R}. \tag{20.6}$$

Replacing the y in this inequality with $-y$ results in

$$|x+y| \leq |x| + |y| \quad \text{for any } x, y \in \mathbb{R}. \tag{20.7}$$

Also by the triangle inequality, $|x-0| \leq |x-(-y)| + |-y-0|$, which yields

$$|x| - |y| \leq |x+y| \quad \text{for any } x, y \in \mathbb{R}. \tag{20.8}$$

The three inequalities (20.6), (20.7) and (20.8) are very useful in proofs. They can be iterated. For example, (20.6) and 20.7) together yield

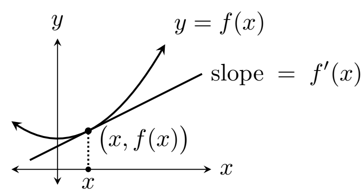
$$|x-y+z| \leq |x| + |y| + |z| \quad \text{for any } x, y, z \in \mathbb{R}, \tag{20.9}$$

and so on. Collectively we call inequalities (20.6)–(20.8) the *triangle inequality*, as they are just variants of Theorem 20.7.

20.6 A Word about Calculus

This text has so far avoided using calculus in an effort to be more self-contained. But this comes at a cost in Chapter 21, where certain proofs and computations that would be very easy with calculus will be more complicated without it. Although we will mostly continue to avoid calculus, certain exercises will be pitched to readers that know it. This brief section reviews two calculus topics relevant in Chapter 21.

Recall that the **derivative** of a function f is another function f' for which $f'(x)$ equals the slope of the tangent line to the graph of $y = f(x)$ at the point $\big(x, f(x)\big)$.



The derivative $f'(x)$ of a function $y = f(x)$ has a variety of notations, including

$$f'(x) \;=\; \frac{d}{dx}\Big[f(x)\Big] \;=\; \frac{dy}{dx}.$$

You learned many derivative rules in your calculus course. To mention just two,

$$\frac{d}{dx}\Big[x^a\Big] \;=\; nx^{n-1} \qquad \text{(power rule)}$$

$$\frac{d}{dx}\Big[\log_a(x)\Big] \;=\; \frac{1}{x\ln(a)}. \qquad \text{(logarithm rule)}$$

Because $f'(x)$ gives the slope of the tangent to $y = f(x)$ at x, it gives information about where f is increasing or decreasing. If $f'(x) > 0$, then the slope at x is positive, so f is increasing at x. If $f'(x) < 0$, then f is decreasing at x.

Fact 20.8. Suppose $f(x)$ is a function that has a derivative $f'(x)$.
- If at some x value, $f'(x) > 0$, then $y = f(x)$ is increasing at x
- If at some x value, $f'(x) < 0$, then $y = f(x)$ is decreasing at x

In your calculus class you learned to apply this fact to find where functions increase and decrease. That process will often be useful in Chapter 21. L'Hôpital's Rule is another calculus topic that can simplify some upcoming computations.

Fact 20.9. L'Hôpital's Rule
If $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ exists and has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \lim\limits_{x\to c} \frac{f'(x)}{g'(x)}$.

Do not be concerned if all this seems unfamiliar, for it is not absolutely essential in Chapter 21. But it is a blunt fact that any serious student of discrete mathematics should know calculus.

Solutions for Chapter 20

Section 20.1

(1) $25^{1/2} = \sqrt{25} = 5$

(2) $\frac{1}{4}^{1/2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

(3) $(-27)^{1/3} = \sqrt[3]{-27} = -3$

(4) $(-27)^{4/3} = \sqrt[3]{-27}^4 = (-3)^4 = 81$

(5) $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

(6) $\frac{1}{2}^{-1} = \frac{1}{\frac{1}{2}} = 2$

(7) $\frac{1}{2}^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8$

(8) $\sqrt{2}^6 = \left(\sqrt{2}^2\right)^3 = 2^3 = 8$

(9) $\left(\frac{\sqrt{3}}{2}\right)^{-4} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^4} = \frac{1}{\frac{9}{16}} = \frac{16}{9}$

(10) $\left(\left(\frac{2}{3}\right)^{\frac{3}{2}}\right)^2 = \left(\sqrt{\frac{2}{3}}^3\right)^2 = \left(\sqrt{\frac{2}{3}}^2\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

(11) $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$

Section 20.4

(1) $\log_3(81) = 3^{\square}(81) = 4$

(2) $\log_3(\sqrt{3}) = 3^{\square}(3^{\frac{1}{2}} = \frac{1}{2}$

(3) $\log_3(1) = 3^{\square}(1) = 0$

(4) $\log(\sqrt[3]{10}) = 10^{\square}(10^{\frac{1}{3}}) = \frac{1}{3}$

(5) $\log(0.01) = 10^{\square}(10^{-2}) = -2$

(6) $\log_4(4) = 4^{\square}(4) = 1$

(7) $\log_4(\sqrt{2}) = \log_4(2^{\frac{1}{2}}) = \frac{1}{2}\log_4(2) = \frac{1}{4}$

(8) $\log_4(8) = \log_4(2^3) = 3\log_4(2) = 3 \cdot \frac{1}{2} = \frac{3}{2}$

(9) $10^{\log(5x+1)} = 5x + 1.$

(10) $\log(2) + \log(5) = \log(2 \cdot 5) = \log(10) = 1$

(11)
$$\log_2(2) - \log_2(5x) + \log_2(20x) = \log_2\left(\frac{2}{5x}\right) + \log_2(20x) = \log_2\left(\frac{40x}{5x}\right) = \log_2(8) = 3$$

(12) $\log_2(\sin(x)) + \dfrac{1}{2}\log_2(4x) - 3\log_2(3) = \log_2(\sin(x)) + \log_2\left((4x)^{\frac{1}{2}}\right) - \log_2(3^3) = \log_2(\sin(x)) + \log_2(2\sqrt{x}) - \log_2(27) = \log_2\left(\frac{2\sqrt{x}\sin(x)}{27}\right).$

(13) $\log(2x) + \log(5x) = \log(2x \cdot 5x) = \log(10x^2) = \log(10) + \log(x^2) = 1 + 2\log(x)$

(14) $\log_2\left((x+5)^4 x^7 \sqrt{x+1}\right) = \log_2(x+5)^4 + \log_2(x^7) + \log_2(\sqrt{x+1}) = 4\log_2(x+5) + 7\log_2(x) + \log_2\left((x+1)^{\frac{1}{2}}\right) = 4\log_2(x+5) + 7\log_2(x) + \frac{1}{2}\log_2(x+1)$

(15) $\log_3\left(\frac{3}{5\sqrt[3]{x}}\right) = \log_3(3) - \log_3(5\sqrt[3]{x}) = 1 - \log_3(5) - \log_3\left(x^{\frac{1}{3}}\right) = 1 - \log_3(5) - \frac{1}{3}\log_3(x)$

(16) $\log_3(5) = \dfrac{\log(5)}{\log(3)} \approx 1.4649$

(17) $\log_5(5) = \dfrac{\log(5)}{\log(5)} = 1$

(18) $\log_9(10) = \dfrac{\log(10)}{\log(9)} \approx 1.0479$

(19) $\log_2(33) = \dfrac{\log(33)}{\log(2)} \approx 5.04438$

(20) $\log_3(9) = \dfrac{\log(9)}{\log(3)} = 2$