Chapter 17

Relations

In mathematics there are endless ways that two entities can be related to each other. Consider the following mathematical statements.

$$\begin{array}{ccccccc} 5 < 10 & 5 \leq 5 & 6 = \frac{30}{5} & 5 \mid 80 & 7 > 4 & x \neq y & 8 \nmid 3 \\ a \equiv b \pmod{n} & 6 \in \mathbb{Z} & X \subseteq Y & \pi \approx 3.14 & 0 \geq -1 & \sqrt{2} \notin \mathbb{Z} & \mathbb{Z} \nsubseteq \mathbb{N} \end{array}$$

In each case two entities appear on either side of a symbol, and we interpret the symbol as expressing some relationship between the two entities. Symbols such as $<, \leq, =, |, \nmid, \geq, >$, $\in$ and $\subset$, etc., are called *relations* because they convey relationships among things.

Relations are significant. In fact, you would have to admit that there would be precious little left of mathematics if we took away all the relations. Therefore it is important to have a firm understanding of them, and this chapter is intended to develop that understanding.

Rather than focusing on each relation individually (an impossible task anyway since there are infinitely many different relations), we will develop a general theory that encompasses *all* relations. Understanding this general theory will give us the conceptual framework and language needed to understand and discuss any specific relation.

17.1 Relations

Before stating the definition of a relation, let's look at a motivational example. This example will lead naturally to our definition.

Consider the set $A = \{1, 2, 3, 4, 5\}$. (There's nothing special about this particular set; any set of numbers would do for this example.) Elements of A can be compared to each other by the symbol "$<$." For example, $1 < 4$, $2 < 3$, $2 < 4$, and so on. You have no trouble understanding this because the notion of numeric order is so ingrained. But imagine you had to explain it to a robot, one with an obsession for detail but absolutely no understanding of the meaning of (or relationships between) integers. You might consider writing down for the robot the following set:

$$R = \{ (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5) \}.$$

The set R encodes the meaning of the $<$ relation for elements in A. An ordered pair (a, b) appears in the set if and only if $a < b$. If asked whether or not it is true that $3 < 4$, the robot could look through R until it found the ordered pair $(3, 4)$; then it would know $3 < 4$ is true. If asked about $5 < 2$, it would see that $(5, 2)$ *does not* appear in R, so $5 \not< 2$. The set R, which is a subset of $A \times A$, completely describes the relation $<$ for A.

Though it may seem simple-minded at first, this is exactly the idea we will use for our main definition. This definition is general enough to describe not just the relation $<$ for the set $A = \{1, 2, 3, 4, 5\}$, but *any* relation for *any* set A.

Definition 17.1. A **relation** on a set A is a subset $R \subseteq A \times A$. We may abbreviate the statement $(x, y) \in R$ as xRy. The statement $(x, y) \notin R$ is abbreviated as $x \not R\, y$.

Notice that a relation is a set, so we can use what we know about sets to understand and explore relations. But before getting deeper into the theory of relations, let's look at some examples of Definition 17.1.

Example 17.1. Let $A = \{1, 2, 3, 4\}$, and consider the following set:

$$R = \{(1,1), (2,1), (2,2), (3,3), (3,2), (3,1), (4,4), (4,3), (4,2), (4,1)\} \subseteq A \times A.$$

The set R is a relation on A, by Definition 17.1. Since $(1, 1) \in R$, we have $1R1$. Similarly $2R1$ and $2R2$, and so on. However, notice that (for example) $(3, 4) \notin R$, so $3 \not R\, 4$. Observe that R is the familiar relation $\geq$ for the set A.

Chapter 1 proclaimed that all of mathematics can be described with sets. Just look at how successful this program has been! The greater-than-or-equal-to relation is now a set R. (We might even express this in the rather cryptic form $\geq = R$.) ✍

Example 17.2. Let $A = \{1, 2, 3, 4\}$, and consider the following set:

$$S = \{(1,1), (1,3), (3,1), (3,3), (2,2), (2,4), (4,2), (4,4)\} \subseteq A \times A.$$

Here we have $1S1$, $1S3$, $4S2$, etc., but $3 \not S\, 4$ and $2 \not S\, 1$. What does S mean? Think of it as meaning "*has the same parity as.*" Thus $1S1$ reads "1 *has the same parity as* 1," and $4S2$ reads "4 *has the same parity as* 2." ✍

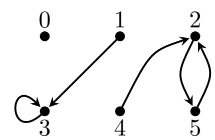
Example 17.3. Consider relations R and S of the previous two examples. Note that $R \cap S = \{(1,1), (2,2), (3,3), (3,1), (4,4), (4,2)\} \subseteq A \times A$ is a relation on A. The expression $x(R \cap S)y$ means "*$x \geq y$, and x, y have the same parity.*" ✍

Example 17.4. Let $B = \{0, 1, 2, 3, 4, 5\}$, and consider the following set:

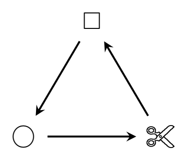
$$U = \{(1,3), (3,3), (5,2), (2,5), (4,2)\} \subseteq B \times B.$$

Then U is a relation on B because $U \subseteq B \times B$. You may be hard-pressed to invent any "meaning" for this particular relation. A relation does not have to have any meaning. Any random subset of $B \times B$ is a relation on B, whether or not it describes anything familiar. ✍

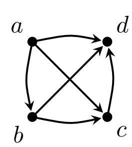
Some relations can be described with pictures. For example, we can depict the above relation U on B by drawing points labeled by elements of B. The statement $(x, y) \in U$ is then represented by an arrow pointing from x to y, a graphic symbol meaning *"x relates to y."* Here's a picture of U:



For another example, the set $A = \big\{\bigcirc, \square, ✄\big\}$ consists of symbols for a rock, a piece of paper, and a pair of scissors. Let R be the relation on A where xRy means x beats y in the children's game Rock-Paper-Scissors. Here is a diagram for R.



The next picture illustrates the relation R on the set $A = \big\{a, b, c, d\big\}$, where xRy means x comes before y in the alphabet. According to Definition 17.1, as a set this relation is $R = \big\{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\big\}$. You may feel that the picture conveys the relation better than the set does. They are two different ways of expressing the same thing. In some instances pictures are more convenient than sets for discussing relations.



Although such diagrams can help us visualize relations, they do have limitations. If A and R were infinite, then the diagram would be impossible to draw, but the set R might be easily expressed in set-builder notation. Here are some examples.

Example 17.5. Consider the set $R = \big\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x - y \in \mathbb{N}\big\} \subseteq \mathbb{Z} \times \mathbb{Z}$. This is the $>$ relation on the set $A = \mathbb{Z}$. It is infinite because there are infinitely many ways to have $x > y$ where x and y are integers. ✍

Example 17.6. The set $R = \big\{(x, x) : x \in \mathbb{R}\big\} \subseteq \mathbb{R} \times \mathbb{R}$ is the relation $=$ on the set $\mathbb{R}$, because xRy means the same thing as $x = y$. Thus R is a set that expresses the notion of equality of real numbers. ✍

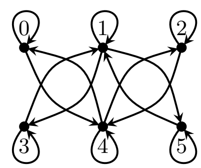
Exercises for Section 17.1

1. Let $A = \{0, 1, 2, 3, 4, 5\}$. Write out the relation R that expresses $>$ on A. Then illustrate it with a diagram.

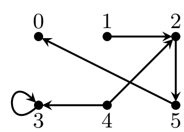
2. Let $A = \{1, 2, 3, 4, 5, 6\}$. Write out the relation R that expresses $\mid$ (divides) on A. Then illustrate it with a diagram.

3. Let $A = \{0, 1, 2, 3, 4, 5\}$. Write out the relation R that expresses $\geq$ on A. Then illustrate it with a diagram.

4. Here is a diagram for a relation R on a set A. Write the sets A and R.



5. Here is a diagram for a relation R on a set A. Write the sets A and R.



6. Congruence modulo 5 is a relation on the set $A = \mathbb{Z}$. In this relation xRy means $x \equiv y \pmod 5$. Write out the set R in set-builder notation.

7. Write the relation $<$ on the set $A = \mathbb{Z}$ as a subset R of $\mathbb{Z} \times \mathbb{Z}$. This is an infinite set, so you will have to use set-builder notation.

8. Let $A = \{1, 2, 3, 4, 5, 6\}$. Observe that $\emptyset \subseteq A \times A$, so $R = \emptyset$ is a relation on A. Draw a diagram for this relation.

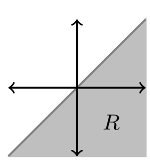
9. Let $A = \{1, 2, 3, 4, 5, 6\}$. How many different relations are there on the set A?

10. Consider the subset $R = (\mathbb{R} \times \mathbb{R}) - \{(x, x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. What familiar relation on $\mathbb{R}$ is this? Explain.

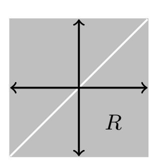
11. Given a finite set A, how many different relations are there on A?

In the following exercises, subsets R of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ or $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ are indicated by gray shading. In each case, R is a familiar relation on $\mathbb{R}$ or $\mathbb{Z}$. State it.

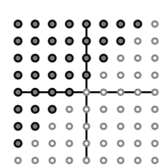
12.



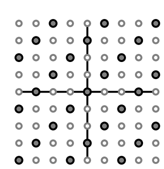
13.



14.



15.



17.2 Properties of Relations

A relational expression xRy is a *statement* (or an *open sentence*); it is either true or false. For example, $5 < 10$ is true, and $10 < 5$ is false. (Thus an operation like $+$ is not a relation, because, for instance, $5+10$ has a numeric value, not a T/F value.) Since relational expressions have T/F values, we can combine them with logical operators; for example, $xRy \Rightarrow yRx$ is a statement or open sentence whose truth or falsity may depend on x and y.

With this in mind, note that some relations have properties that others don't have. For example, the relation $\leq$ on $\mathbb{Z}$ satisfies $x \leq x$ for every $x \in \mathbb{Z}$. But this is not so for $<$ because $x < x$ is never true. The next definition lays out three particularly significant properties that relations may have.

Definition 17.2. Suppose R is a relation on a set A. Then:

- R is **reflexive** if xRx for every $x \in A$.
  That is, R is reflexive if $\forall\, x \in A,\ xRx$.
- R is **symmetric** if xRy implies yRx for all $x, y \in A$
  That is, R is symmetric if $\forall\, x, y \in A,\ xRy \ \Rightarrow\ yRx$.
- R is **transitive** if whenever xRy and yRz, then also xRz.
  That is, R is transitive if $\forall\, x, y, z \in A, \big((xRy) \wedge (yRz)\big) \Rightarrow xRz$.

To illustrate this, let's consider the set $A = \mathbb{Z}$. Examples of reflexive relations on $\mathbb{Z}$ include $\leq, =$, and $|$, because $x \leq x$, $x = x$ and $x \mid x$ are all true for any $x \in \mathbb{Z}$. On the other hand, $>, <, \neq$ and $\nmid$ are not reflexive for none of the statements $x < x$, $x > x$, $x \neq x$ and $x \nmid x$ is ever true.

The relation $\neq$ **is** symmetric, for if $x \neq y$, then surely $y \neq x$ also. Also, the relation $=$ is symmetric because $x = y$ always implies $y = x$.

The relation $\leq$ is **not** symmetric, as $x \leq y$ does not necessarily imply $y \leq x$. For instance $5 \leq 6$ is true, but $6 \leq 5$ is false. Notice $(x \leq y) \ \Rightarrow\ (y \leq x)$ *is* true for some x and y (for example, it is true when $x = 2$ and $y = 2$), but still $\leq$ is not symmetric because it is not the case that $(x \leq y) \ \Rightarrow\ (y \leq x)$ is true for *all* integers x and y.

The relation $\leq$ is transitive because whenever $x \leq y$ and $y \leq z$, it also is true that $x \leq z$. Likewise $<, \geq, >$ and $=$ are all transitive. Examine the following table and be sure you understand why it is labeled as it is.

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive	no	yes	yes	yes	no	no
Symmetric	no	no	yes	no	no	yes
Transitive	yes	yes	yes	yes	no	no

Example 17.7. Consider the set $A = \{b, c, d, e\}$, and let R be the following relation on A: $R = \{(b, b), (b, c), (c, b), (c, c), (d, d), (b, d), (d, b), (c, d), (d, c)\}$. Is R is reflexive? Is it symmetric? Is it transitive?.

Solution: This relation is **not** reflexive, for although bRb, cRc and dRd, it is **not** true that eRe. For a relation to be reflexive, xRx must be true for *all* $x \in A$.

The relation R **is** symmetric, because whenever we have xRy, it follows that yRx too. Observe that bRc and cRb; bRd and dRb; dRc and cRd. Take away the ordered pair (c, b) from R, and R is no longer symmetric.

The relation R is transitive, but it takes some work to check it. We must check that the statement $(xRy \wedge yRz) \Rightarrow xRz$ is true for all $x, y, z \in A$. For example, taking $x = b$, $y = c$ and $z = d$, we have $(bRc \wedge cRd) \Rightarrow bRd$, which is the true statement $(T \wedge T) \Rightarrow T$. Likewise, $(bRd \wedge dRc) \Rightarrow bRc$ is the true statement $(T \wedge T) \Rightarrow T$. Take note that if $x = b$, $y = e$ and $z = c$, then $(bRe \wedge eRc) \Rightarrow bRc$ becomes $(F \wedge F) \Rightarrow T$, which is *still* true. It's not much fun, but going through all the combinations, you can verify that $(xRy \wedge yRz) \Rightarrow xRz$ is true for all choices $x, y, z \in A$. (Try at least a few of them.) ✍

The relation R from Example 17.7 has a meaning. You can think of xRy as meaning that x and y are both consonants. Thus bRc because b and c are both consonants; but $b \not R e$ because it's not true that b and e are both consonants. Once we look at it this way, it's immediately clear that R has to be transitive. If x and y are both consonants and y and z are both consonants, then surely x and z are both consonants. This illustrates a point that we will see again later in this section: Knowing the meaning of a relation can help us understand it and prove things about it.

Here is a picture of R. Notice that we can immediately spot several properties of R that may not have been so clear from its set description. For instance, we see that R is not reflexive because it lacks a loop at e, hence $e \not R e$.

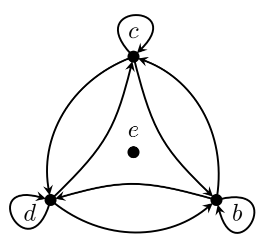


Fig. 17.1 The relation R from Example 17.7

In what follows, we summarize how to spot the various properties of a relation from its diagram. Compare these with Figure 17.2.

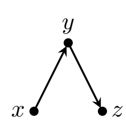
1. A relation is **reflexive** if for each point x ...

...there is a loop at x:

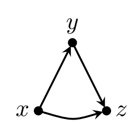
2. A relation is **symmetric** if whenever there is an arrow from x to y ...

...there is also an arrow from y back to x:

3. A relation is **transitive** if whenever there are arrows from x to y and y to z ...



...there is also an arrow from x to z:



(If $x = z$, this means that if there are arrows from x to y and from y to x ...

...there is also a loop from x back to x.)

Consider the bottom diagram in Box 3, above. The transitive property demands $(xRy \wedge yRx) \Rightarrow xRx$. Thus, if xRy and yRx in a transitive relation, then also xRx, so there is a loop at x. In this case $(yRx \wedge xRy) \Rightarrow yRy$, so there will be a loop at y too.

Although these visual aids can be illuminating, their use is limited because many relations are too large and complex to be adequately described as diagrams. For example, it would be impossible to draw a diagram for the relation $\equiv \pmod{n}$, where $n \in \mathbb{N}$. Such a relation would best be explained in a more theoretical (and less visual) way.

We next prove that $\equiv \pmod{n}$ is reflexive, symmetric and transitive. Obviously we will not glean this from a drawing. Instead we will prove it from the properties of $\equiv \pmod{n}$ and Definition 17.2. Pay attention to this example. It illustrates how to **prove** things about relations.

Example 17.8. Prove the following proposition.

Proposition Let $n \in \mathbb{N}$. The relation $\equiv \pmod{n}$ on the set $\mathbb{Z}$ is reflexive, symmetric and transitive.

Proof. First we will show that $\equiv \pmod{n}$ is reflexive. Take any integer $x \in \mathbb{Z}$, and observe that $n \mid 0$, so $n \mid (x - x)$. By definition of congruence modulo n, we have $x \equiv x \pmod{n}$. This shows $x \equiv x \pmod{n}$ for every $x \in \mathbb{Z}$, so $\equiv \pmod{n}$ is reflexive.

Next, we will show that $\equiv \pmod{n}$ is symmetric. For this, we must show that for all $x, y \in \mathbb{Z}$, the condition $x \equiv y \pmod{n}$ implies that $y \equiv x \pmod{n}$. We use direct proof. Suppose $x \equiv y \pmod{n}$. Thus $n \mid (x - y)$ by definition of congruence modulo n. Then $x - y = na$ for some $a \in \mathbb{Z}$ by definition of divisibility. Multiplying both sides by -1 gives $y - x = n(-a)$. Therefore $n \mid (y - x)$, and this means $y \equiv x \pmod{n}$. We've shown that $x \equiv y \pmod{n}$ implies that $y \equiv x \pmod{n}$, and this means $\equiv \pmod{n}$ is symmetric.

Finally we will show that $\equiv \pmod{n}$ is transitive. For this we must show that if $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$, then $x \equiv z \pmod{n}$. Again we use direct proof. Suppose $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$. This means $n \mid (x - y)$ and $n \mid (y - z)$. Therefore there are integers a and b for which $x - y = na$ and $y - z = nb$. Adding these two equations, we obtain $x - z = na + nb$. Consequently, $x - z = n(a + b)$, so $n \mid (x - z)$, hence $x \equiv z \pmod{n}$. This completes the proof that $\equiv \pmod{n}$ is transitive.

The past three paragraphs have shown that $\equiv \pmod{n}$ is reflexive, symmetric and transitive, so the proof is complete. $\square$

As you continue with mathematics you will find that the reflexive, symmetric and transitive properties take on special significance in a variety of settings. In preparation for this, the next section explores further consequences of these properties. But first work some of the following exercises.

Exercises for Section 17.2

1. Consider the relation $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$ on $A = \{a,b,c,d\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

2. Consider the relation $R = \{(a,b),(a,c),(c,c),(b,b),(c,b),(b,c)\}$ on $A = \{a,b,c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

3. Consider the relation $R = \{(a,b),(a,c),(c,b),(b,c)\}$ on $A = \{a,b,c\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

4. Let $A = \{a,b,c,d\}$. Suppose R is the relation

$$\begin{aligned} R = \{&(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(c,a),\\ &(a,d),(d,a),(b,c),(c,b),(b,d),(d,b),(c,d),(d,c)\}. \end{aligned}$$

Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

5. Consider the relation $R = \{(0,0), (\sqrt{2}, 0), (0, \sqrt{2}), (\sqrt{2}, \sqrt{2})\}$ on $\mathbb{R}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

6. Consider the relation $R = \{(x,x) : x \in \mathbb{Z}\}$ on $\mathbb{Z}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?

7. There are 16 possible different relations R on the set $A = \{a, b\}$. Describe all of them. (A picture for each one will suffice, but don't forget to label the nodes.) Which ones are reflexive? Symmetric? Transitive?

8. Define a relation on $\mathbb{Z}$ as xRy if $|x - y| < 1$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?

9. Define a relation on $\mathbb{Z}$ by declaring xRy if and only if x and y have the same parity. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?

10. Suppose $A \neq \emptyset$. Since $\emptyset \subseteq A \times A$, the set $R = \emptyset$ is a relation on A. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

11. Suppose $A = \{a, b, c, d\}$ and $R = \{(a,a), (b,b), (c,c), (d,d)\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why.

12. Prove that the relation $|$ (divides) on the set $\mathbb{Z}$ is reflexive and transitive. (Use Example 17.8 as a guide if you are unsure of how to proceed.)

13. Consider the relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$. Prove that this relation is reflexive, symmetric and transitive.

14. Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.

15. Prove or disprove: If a relation is symmetric and transitive, then it is also reflexive.

16. Define a relation R on $\mathbb{Z}$ by declaring that xRy if and only if $x^2 \equiv y^2 \pmod{4}$. Prove that R is reflexive, symmetric and transitive.

17. Modifying the above Exercise 8 (above) slightly, define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $|x - y| \leq 1$. Say whether $\sim$ is reflexive. Is it symmetric? Transitive?

18. The table on page 377 shows that relations on $\mathbb{Z}$ may obey various combinations of the reflexive, symmetric and transitive properties. In all, there are $2^3 = 8$ possible combinations, and the table shows 5 of them. (There is some redundancy, as $\leq$ and $|$ have the same type.) Complete the table by finding examples of relations on $\mathbb{Z}$ for the three missing combinations.

17.3 Equivalence Relations

The relation $=$ on the set $\mathbb{Z}$ (or on any set A) is reflexive, symmetric and transitive. There are many other relations that are also reflexive, symmetric and transitive. Relations that have all three of these properties occur very frequently in mathematics and often play quite significant roles. (For instance, this is certainly true of the relation $=$.) Such relations are given a special name. They are called *equivalence relations*.

Definition 17.3. A relation R on a set A is an **equivalence relation** if it is reflexive, symmetric and transitive.

As an example, Figure 17.2 shows four different equivalence relations R_1, R_2, R_3 and R_4 on the set $A = \{-1, 1, 2, 3, 4\}$. Each one has its own meaning, as labeled. For example, in the second row the relation R_2 literally means *"has the same parity as."* So $1 R_2\, 3$ means "1 *has the same parity as* 3," etc.

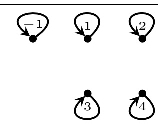
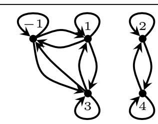
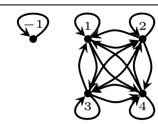
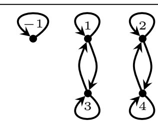
Relation R	Diagram	Equivalence classes (see next page)
"is equal to" $(=)$ $R_1 = \{(-1,-1), (1,1), (2,2), (3,3), (4,4)\}$		$\{-1\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$
"has same parity as" $R_2 = \{(-1,-1), (1,1), (2,2), (3,3), (4,4),$ $(-1,1), (1,-1), (-1,3), (3,-1),$ $(1,3), (3,1), (2,4), (4,2)\}$		$\{-1, 1, 3\}$, $\{2, 4\}$
"has same sign as" $R_3 = \{(-1,-1), (1,1), (2,2), (3,3), (4,4),$ $(1,2), (2,1), (1,3), (3,1), (1,4), (4,1),$ $(2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}$		$\{-1\}$, $\{1, 2, 3, 4\}$
"has same parity and sign as" $R_4 = \{(-1,-1), (1,1), (2,2), (3,3),$ $(4,4), (1,3), (3,1), (2,4), (4,2)\}$		$\{-1\}$, $\{1, 3\}$, $\{2, 4\}$

Fig. 17.2 Examples of equivalence relations on the set $A = \{-1, 1, 2, 3, 4\}$

The above diagrams make it easy to check that each relation is reflexive, symmetric and transitive, i.e., that each is an equivalence relation. For example, R_1 is symmetric because $xR_1y \Rightarrow yR_1x$ is always true: When $x = y$ it becomes $T \Rightarrow T$ (true), and when $x \neq y$ it becomes $F \Rightarrow F$ (also true). In a similar fashion, R_1 is transitive because $(xR_1y \wedge yR_1z) \Rightarrow xR_1z$ is always true: It always works out to one of $T \Rightarrow T$, $F \Rightarrow T$ or $F \Rightarrow F$. (Check this.)

As you can see from the examples in Figure 17.2, equivalence relations on a set tend to express some measure of "sameness" among the elements of the set, whether it is true equality or something weaker (like having the same parity).

It's time to introduce an important definition. Whenever you have an equivalence relation R on a set A, it divides A into subsets called *equivalence classes*. Here is the definition:

Definition 17.4. Suppose R is an equivalence relation on a set A. Given any element $a \in A$, the **equivalence class containing** a is the subset $\{x \in A : xRa\}$ of A consisting of all the elements of A that relate to a. This set is denoted as $[a]$. Thus the equivalence class containing a is the set $[a] = \{x \in A : xRa\}$.

Example 17.9. Consider the relation R_1 in Figure 17.2. The equivalence class containing 2 is the set $[2] = \{x \in A : xR_1 2\}$. Because in this relation the only element that relates to 2 is 2 itself, we have $[2] = \{2\}$. Other equivalence classes for R_1 are $[-1] = \{-1\}$, $[1] = \{1\}$, $[3] = \{3\}$ and $[4] = \{4\}$. Thus this relation has five separate equivalence classes. ✍

Example 17.10. Consider the relation R_2 in Figure 17.2. The equivalence class containing 2 is the set $[2] = \{x \in A : xR_2 2\}$. Because only 2 and 4 relate to 2, we have $[2] = \{2, 4\}$. Observe that we also have $[4] = \{x \in A : xR_2 4\} = \{2, 4\}$, so $[2] = [4]$. Another equivalence class for R_2 is $[1] = \{x \in A : xR_2 1\} = \{-1, 1, 3\}$. In addition, note that $[1] = [-1] = [3] = \{-1, 1, 3\}$. Thus this relation has just two equivalence classes, namely $\{2, 4\}$ and $\{-1, 1, 3\}$. ✍

Example 17.11. The relation R_4 in Figure 17.2 has three equivalence classes. They are $[-1] = \{-1\}$ and $[1] = [3] = \{1, 3\}$ and $[2] = [4] = \{2, 4\}$. ✍

Don't be misled by Figure 17.2. It's important to realize that not every equivalence relation can be drawn as a diagram involving nodes and arrows. Even the simple relation $R = \{(x, x) : x \in \mathbb{R}\}$, which expresses equality in the set $\mathbb{R}$, is too big to be drawn. Its picture would involve a point for every real number and a loop at each point. Clearly that's too many points and loops to draw.

We close this section with several other examples of equivalence relations on infinite sets.

Example 17.12. Let P be the set of all polynomials with real coefficients. Define a relation R on P as follows. Given $f(x), g(x) \in P$, let $f(x)\,R\,g(x)$ mean that $f(x)$ and $g(x)$ have the same degree. Thus $(x^2 + 3x - 4)\,R\,(3x^2 - 2)$ and $(x^3 + 3x^2 - 4) \not R\,(3x^2 - 2)$, for example. It takes just a quick mental check to see that R is an equivalence relation. (Do it.) It's easy to describe the equivalence classes of R. For example, $[3x^2 + 2]$ is the set of all polynomials that have the same degree as $3x^2 + 2$, that is, the set of all polynomials of degree 2. We can write this as $[3x^2 + 2] = \{ax^2 + bx + c : a, b, c \in \mathbb{R}, a \neq 0\}$. ✍

Example 17.13. Example 17.8 proved that for a given $n \in \mathbb{N}$ the relation $\equiv \pmod{n}$ is reflexive, symmetric and transitive. Thus, in our new parlance, $\equiv \pmod{n}$ is an equivalence relation on $\mathbb{Z}$. Consider the case $n = 3$. Let's find the equivalence classes of the equivalence relation $\equiv \pmod 3$. The equivalence class containing 0 seems like a reasonable place to start. Observe that

$$[0] = \{x \in \mathbb{Z} : x \equiv 0 \pmod 3\} = \\ \{x \in \mathbb{Z} : 3 \mid (x - 0)\} = \{x \in \mathbb{Z} : 3 \mid x\} = \{\ldots, -3, 0, 3, 6, 9, \ldots\}.$$

Thus the class [0] consists of all the multiples of 3. (Or, said differently, [0] consists of all integers that have a remainder of 0 when divided by 3). Note that $[0] = [3] = [6] = [9]$, etc. The number 1 does not show up in the set [0] so let's next look at the equivalence class [1]:

$$[1] = \{x \in \mathbb{Z} : x \equiv 1 \pmod 3\} = \{x \in \mathbb{Z} : 3 \mid (x-1)\} = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}.$$

The equivalence class [1] consists of all integers that give a remainder of 1 when divided by 3. The number 2 is in neither of the sets [0] or [1], so we next look at the equivalence class [2]:

$$[2] = \{x \in \mathbb{Z} : x \equiv 2 \pmod 3\} = \{x \in \mathbb{Z} : 3 \mid (x-2)\} = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}.$$

The equivalence class [2] consists of all integers that give a remainder of 2 when divided by 3. Observe that any integer is in one of the sets [0], [1] or [2], so we have listed all of the equivalence classes. Thus $\equiv \pmod 3$ has exactly three equivalence classes, as described above.

Similarly, you can show that the equivalence relation $\equiv \pmod n$ has n equivalence classes $[0], [1], [2], \ldots, [n-1]$. ✍

The idea of an equivalence relation is fundamental. In a very real sense you have dealt with equivalence relations for much of your life, without being aware of it. In fact your conception of fractions is entwined with an intuitive notion of an equivalence relation. To see how this is so, consider the set of all fractions, *not necessarily reduced*:

$$F = \left\{ \frac{m}{n} \;:\; m, n \in \mathbb{Z}, n \neq 0 \right\}.$$

Interpret this set not as $\mathbb{Q}$, but rather as the set of all possible fractions. For example, we consider the fractions $\frac{1}{2}$ and $\frac{2}{4}$ as being distinct (unequal) elements of F because their numerators and denominators don't match. Of course $\frac{1}{2}$ and $\frac{2}{4}$ are equal numbers, but they are *different* fractions, so $\frac{1}{2}, \frac{2}{4} \in F$, but $\frac{1}{2} \neq \frac{2}{4}$ (meaning they are distinct, unequal elements of F).

Define a relation $\doteq$ on F by saying $\frac{a}{b} \doteq \frac{c}{d}$ provided that $ad = bc$. Thus $\frac{1}{2} \doteq \frac{2}{4}$ because $1 \cdot 4 = 2 \cdot 2$. Similarly, notice that $\frac{-15}{-3} \doteq \frac{10}{2}$ because $-15 \cdot 2 = -3 \cdot 10$. We have defined $\doteq$ so that $\frac{a}{b} \doteq \frac{c}{d}$ if and only if $\frac{a}{b}$ and $\frac{c}{d}$ are equal numbers, so $\doteq$ models your intuitive, ingrained understanding of when two different fractions are equal.

Observe that $\doteq$ is an equivalence relation on the set F of all fractions: It is reflexive because for any $\frac{a}{b} \in F$ the equation $ab = ba$ guarantees $\frac{a}{b} \doteq \frac{a}{b}$. To see that $\doteq$ is symmetric, suppose $\frac{a}{b} \doteq \frac{c}{d}$. This means $ad = bc$, so $cb = da$, which implies $\frac{c}{d} \doteq \frac{a}{b}$. Exercise 16 below asks you to confirm that $\doteq$ is transitive.

This discussion shows that your everyday understanding of equality of fractions is an equivalence relation. The equivalence class containing, say, $\frac{2}{3}$ is the set $\left\{\frac{2n}{3n} : n \in \mathbb{Z}, n \neq 0\right\}$ of all fractions that are numerically equal to $\frac{2}{3}$. The takeaway is that you have for years lumped together equal fractions into equivalence classes under this equivalence relation.

This undersores an important point: Equivalence relations arise in many areas of mathematics. This is especially true in the advanced realms of mathematics, where equivalence relations are the right tool for important constructions, constructions as natural and far-reaching as fractions. Learning about equivalence relations now paves the way to a deeper understanding of later courses, and work.

Exercises for Section 17.3

1. Let $A = \{1, 2, 3, 4, 5, 6\}$, and consider the following equivalence relation on A: $R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(2,3),(3,2),(4,5),(5,4),(4,6),(6,4),(5,6),(6,5)\}$. List the equivalence classes of R.

2. Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose R has two equivalence classes. Also aRd, bRc and eRd. Write out R as a set.

3. Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.

4. Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose also that aRd and bRc, eRa and cRe. How many equivalence classes does R have?

5. There are two different equivalence relations on the set $A = \{a, b\}$. Describe them. Diagrams will suffice.

6. There are five different equivalence relations on the set $A = \{a, b, c\}$. Describe them all. Diagrams will suffice.

7. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x - 5y$ is even. Prove R is an equivalence relation. Describe its equivalence classes.

8. Define a relation R on $\mathbb{Z}$ as xRy if and only if $x^2 + y^2$ is even. Prove R is an equivalence relation. Describe its equivalence classes.

9. Define a relation R on $\mathbb{Z}$ as xRy if and only if $4 \mid (x+3y)$. Prove R is an equivalence relation. Describe its equivalence classes.

10. Suppose R and S are two equivalence relations on a set A. Prove that $R \cap S$ is also an equivalence relation. (For an example of this, look at Figure 17.2. Observe that for the equivalence relations R_2, R_3 and R_4, we have $R_2 \cap R_3 = R_4$.)

11. Prove or disprove: If R is an equivalence relation on an infinite set A, then R has infinitely many equivalence classes.

12. Prove or disprove: If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

13. Suppose R is an equivalence relation on a finite set A, and every equivalence class has the same cardinality m. Express $|R|$ in terms of m and $|A|$.

14. Suppose R is a reflexive and symmetric relation on a finite set A. Define a relation S on A by declaring xSy if and only if for some $n \in \mathbb{N}$ there are elements $x_1, x_2, \ldots, x_n \in A$ satisfying $xRx_1,\ x_1Rx_2,\ x_2Rx_3,\ x_3Rx_4, \ldots, x_{n-1}Rx_n$, and x_nRy. Show that S is an equivalence relation and $R \subseteq S$. Prove that S is the unique smallest equivalence relation on A containing R.

15. Suppose R is an equivalence relation on a set A, with four equivalence classes. How many different equivalence relations S on A are there for which $R \subseteq S$?

16. Show that the relation $\doteq$ defined on page 384 is transitive.

17.4 Equivalence Classes and Partitions

This section collects several properties of equivalence classes.

Our first result proves that $[a] = [b]$ if and only if aRb. This is useful because it assures us that whenever we are in a situation where $[a] = [b]$, we also have aRb, and vice versa. Being able to switch back and forth between these two pieces of information can be helpful in a variety of situations, and you may find yourself using this result a lot. Be sure to notice that the proof uses all three properties (reflexive, symmetric and transitive) of equivalence relations. Notice also that we have to use some Chapter 12 techniques in dealing with the sets $[a]$ and $[b]$.

Theorem 17.1. *Suppose R is an equivalence relation on a set A. Suppose also that $a, b \in A$. Then $[a] = [b]$ if and only if aRb.*

Proof. Suppose $[a] = [b]$. Note that aRa by the reflexive property of R, so $a \in \{x \in A : xRa\} = [a] = [b] = \{x \in A : xRb\}$. But a belonging to $\{x \in A : xRb\}$ means aRb. This completes the first part of the if-and-only-if proof.

Conversely, suppose aRb. We need to show $[a] = [b]$. We will do this by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

First we show $[a] \subseteq [b]$. Suppose $c \in [a]$. As $c \in [a] = \{x \in A : xRa\}$, we get cRa. Now we have cRa and aRb, so cRb because R is transitive. But cRb implies $c \in \{x \in A : xRb\} = [b]$. This demonstrates that $c \in [a]$ implies $c \in [b]$, so $[a] \subseteq [b]$.

Next we show $[b] \subseteq [a]$. Suppose $c \in [b]$. As $c \in [b] = \{x \in A : xRb\}$, we get cRb. Remember that we are assuming aRb, so bRa because R is symmetric. Now we have cRb and bRa, so cRa because R is transitive. But cRa implies $c \in \{x \in A : xRa\} = [a]$. This demonstrates that $c \in [b]$ implies $c \in [a]$; hence $[b] \subseteq [a]$.

The previous two paragraphs imply that $[a] = [b]$. $\square$

To illustrate Theorem 17.1, recall how we worked out the equivalence classes of $\equiv \pmod 3$ at the end of Section 17.3. We observed that

$$[-3] = [9] = \{\dots, -3, 0, 3, 6, 9, \dots\}.$$

Note that $[-3] = [9]$ and $-3 \equiv 9 \pmod 3$, just as Theorem 17.1 predicts. The theorem assures us that this will work for any equivalence relation. In the future you may find yourself using the result of Theorem 17.1 often. Over time it may become natural and familiar; you will use it automatically, without even thinking of it as a theorem.

Our next topic addresses the fact that an equivalence relation on a set A divides A into various equivalence classes. There is a special word for this kind of situation. We address it now, as you are likely to encounter it in subsequent mathematics classes.

Definition 17.5. A **partition** of a set A is a set of non-empty subsets of A, such that the union of all the subsets equals A, and the intersection of any two different subsets is $\emptyset$.

Example 17.14. Let $A = \big\{a,b,c,d\big\}$. One partition of A is $\big\{\{a,b\},\{c\},\{d\}\big\}$. This is a set of three subsets $\{a,b\}$, $\{c\}$ and $\{d\}$ of A. The union of the three subsets equals A; the intersection of any two subsets is $\emptyset$.

Other partitions of A are

$$\big\{\{a,b\},\{c,d\}\big\},\ \big\{\{a,c\},\{b\},\{d\}\big\},\ \big\{\{a\},\{b\},\{c\}\{d\}\big\},\ \big\{\{a,b,c,d\}\big\},$$

to name a few. Intuitively, a partition is just a dividing of A into parts. ✍

Example 17.15. Consider the equivalence relations in Figure 17.2. Each of these is a relation on the set $A = \big\{-1,1,2,3,4\big\}$. The equivalence classes of each relation are listed on the right side of the figure. Observe that, in each case, the set of equivalence classes forms a partition of A. For example, the relation R_1 yields the partition $\big\{\{-1\},\{1\},\{2\},\{3\},\{4\}\big\}$ of A. Also, the equivalence classes of R_2 form the partition $\big\{\{-1,1,3\},\{2,4\}\big\}$. ✍

Example 17.16. Recall that we worked out the equivalence classes of the equivalence relation $\equiv \pmod 3$ on the set $\mathbb{Z}$. These equivalence classes give the following partition of $\mathbb{Z}$:

$$\big\{\{\ldots,-3,0,3,6,9,\ldots\},\{\ldots,-2,1,4,7,10,\ldots\},\{\ldots,-1,2,5,8,11,\ldots\}\big\}.$$

We can write it more compactly as $\big\{[0],[1],[2]\big\}$. ✍

Our examples and experience suggest that the equivalence classes of an equivalence relation on a set form a partition of that set. This is indeed the case, and we now prove it.

Theorem 17.2. *Suppose R is an equivalence relation on a set A. Then the set $\big\{[a] : a \in A\big\}$ of equivalence classes of R forms a partition of A.*

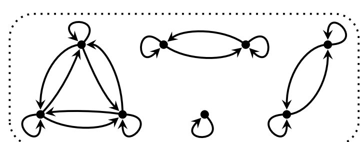
Proof. To show that $\big\{[a] : a \in A\big\}$ is a partition of A we need to show two things: We need to show that the union of all the sets $[a]$ equals A, and we need to show that if $[a] \neq [b]$, then $[a] \cap [b] = \emptyset$.

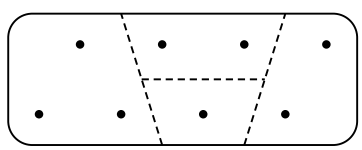
Notationally, the union of all the sets $[a]$ is $\bigcup_{a\in A}[a]$, so we need to prove $\bigcup_{a\in A}[a] = A$. Suppose $x \in \bigcup_{a\in A}[a]$. This means $x \in [a]$ for some $a \in A$. Since $[a] \subseteq A$, it then follows that $x \in A$. Thus $\bigcup_{a\in A}[a] \subseteq A$. On the other hand, suppose $x \in A$. As $x \in [x]$, we know $x \in [a]$ for some $a \in A$ (namely $a = x$). Therefore $x \in \bigcup_{a\in A}[a]$, and this shows $A \subseteq \bigcup_{a\in A}[a]$. Since $\bigcup_{a\in A}[a] \subseteq A$ and $A \subseteq \bigcup_{a\in A}[a]$, it follows that $\bigcup_{a\in A}[a] = A$.

Next we need to show that if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$. Let's use contrapositive proof. Suppose it's not the case that $[a] \cap [b] = \emptyset$, so there is some element c with $c \in [a] \cap [b]$. Thus $c \in [a]$ and $c \in [b]$. Now, $c \in [a]$ means cRa, and then aRc since R is symmetric. Also $c \in [b]$ means cRb. Now we have aRc and cRb, so aRb (because R is transitive). By Theorem 17.1, aRb implies $[a] = [b]$. Thus $[a] \neq [b]$ is not true.

We've now shown that the union of all the equivalence classes is A, and the intersection of two different equivalence classes is $\emptyset$. Therefore the set of equivalence classes is a partition of A. □

Theorem 17.2 says the equivalence classes of any equivalence relation on a set A form a partition of A. Conversely, any partition of A describes an equivalence relation R where xRy if and only if x and y belong to the same set in the partition. (See Exercise 4 for this section, below.) Thus equivalence relations and partitions are really just two different ways of looking at the same thing. In the diagrams below, the equivalence relation on A (top) corresponds to a partition of A (bottom). These are two different ways of expressing the same basic information. You may find yourself switching between these two points of view often.

A  Equivalence relation on A

A  Partition of A

Exercises for Section 17.4

1. List all the partitions of the set $A = \{a, b\}$. Compare your answer to the answer to Exercise 5 of Section 17.3.

2. List all the partitions of the set $A = \{a, b, c\}$. Compare your answer to the answer to Exercise 6 of Section 17.3.

3. Describe the partition of $\mathbb{Z}$ resulting from the equivalence relation $\equiv \pmod 4$.

4. Suppose P is a partition of a set A. Define a relation R on A by declaring xRy if and only if $x, y \in X$ for some $X \in P$. Prove R is an equivalence relation on A. Then prove that P is the set of equivalence classes of R.

5. Consider the partition $P = \big\{\{\dots, -4, -2, 0, 2, 4, \dots\}, \{\dots, -5, -3, -1, 1, 3, 5, \dots\}\big\}$ of $\mathbb{Z}$. Let R be the equivalence relation whose equivalence classes are the two elements of P. What familiar equivalence relation is R?

17.5 The Integers Modulo n

Example 17.8 proved that for a given $n \in \mathbb{N}$, the relation $\equiv \pmod{n}$ is reflexive, symmetric and transitive, so it is an equivalence relation. This is a particularly significant equivalence relation in mathematics, and in the present section we deduce some of its properties.

To make matters simpler, let's pick a concrete n, say $n = 5$. Let's begin by looking at the equivalence classes of the relation $\equiv \pmod{5}$. There are five equivalence classes, as follows:

$$
\begin{aligned}
[0] &= \big\{x \in \mathbb{Z} : x \equiv 0 \ (\mathrm{mod}\ 5))\big\} = \big\{x \in \mathbb{Z} : 5 \mid (x-0)\big\} = \big\{\ldots, -10, -5, 0, 5, 10, 15, \ldots\big\},\\
[1] &= \big\{x \in \mathbb{Z} : x \equiv 1 \ (\mathrm{mod}\ 5))\big\} = \big\{x \in \mathbb{Z} : 5 \mid (x-1)\big\} = \big\{\ldots, \ -9, -4, 1, 6, 11, 16, \ldots\big\},\\
[2] &= \big\{x \in \mathbb{Z} : x \equiv 2 \ (\mathrm{mod}\ 5))\big\} = \big\{x \in \mathbb{Z} : 5 \mid (x-2)\big\} = \big\{\ldots, \ -8, -3, 2, 7, 12, 17, \ldots\big\},\\
[3] &= \big\{x \in \mathbb{Z} : x \equiv 3 \ (\mathrm{mod}\ 5))\big\} = \big\{x \in \mathbb{Z} : 5 \mid (x-3)\big\} = \big\{\ldots, \ -7, -2, 3, 8, 13, 18, \ldots\big\},\\
[4] &= \big\{x \in \mathbb{Z} : x \equiv 4 \ (\mathrm{mod}\ 5))\big\} = \big\{x \in \mathbb{Z} : 5 \mid (x-4)\big\} = \big\{\ldots, \ -6, -1, 4, 9, 14, 19, \ldots\big\}.
\end{aligned}
$$

Notice how these equivalence classes form a partition of the set $\mathbb{Z}$. We label the five equivalence classes as $[0], [1], [2], [3]$ and $[4]$, but you know of course that there are other ways to label them. For example, $[0] = [5] = [10] = [15]$, and so on; and $[1] = [6] = [-4]$, etc. Still, for this discussion we denote the five classes as $[0], [1], [2], [3]$ and $[4]$.

These five classes form a set, which we shall denote as $\mathbb{Z}_5$. Thus

$$\mathbb{Z}_5 = \big\{[0], [1], [2], [3], [4]\big\}$$

is a set of five sets. The interesting thing about $\mathbb{Z}_5$ is that even though its elements are sets (and not numbers), it is possible to add and multiply them. In fact, we can define the following rules that tell how elements of $\mathbb{Z}_5$ can be added and multiplied.

$$
\begin{aligned}
[a] + [b] &= [a + b]\\
[a] \cdot [b] &= [\, a \cdot b\,]
\end{aligned}
$$

For example, $[2] + [1] = [2+1] = [3]$, and $[2] \cdot [2] = [2 \cdot 2] = [4]$. We stress that in doing this we are adding and multiplying *sets* (more precisely equivalence classes), not numbers. We added (or multiplied) two elements of $\mathbb{Z}_5$ and obtained another element of $\mathbb{Z}_5$.

Here is a trickier example. Observe that $[2] + [3] = [5]$. This time we added elements $[2], [3] \in \mathbb{Z}_5$, and got the element $[5] \in \mathbb{Z}_5$. That was easy, except where is our answer $[5]$ in the set $\mathbb{Z}_5 = \big\{[0], [1], [2], [3], [4]\big\}$? Since $[5] = [0]$, it is more appropriate to write $[2] + [3] = [0]$.

In a similar vein, $[2] \cdot [3] = [6]$ would be written as $[2] \cdot [3] = [1]$ because $[6] = [1]$. Test your skill with this by verifying the following addition and multiplication tables for $\mathbb{Z}_5$.

$+$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[0]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[1]$	$[1]$	$[2]$	$[3]$	$[4]$	$[0]$
$[2]$	$[2]$	$[3]$	$[4]$	$[0]$	$[1]$
$[3]$	$[3]$	$[4]$	$[0]$	$[1]$	$[2]$
$[4]$	$[4]$	$[0]$	$[1]$	$[2]$	$[3]$

$\cdot$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$
$[2]$	$[0]$	$[2]$	$[4]$	$[1]$	$[3]$
$[3]$	$[0]$	$[3]$	$[1]$	$[4]$	$[2]$
$[4]$	$[0]$	$[4]$	$[3]$	$[2]$	$[1]$

We call the set $\mathbb{Z}_5 = \big\{[0],[1],[2],[3],[4]\big\}$ the **integers modulo 5**. As our tables suggest, $\mathbb{Z}_5$ is more than just a set: It is a little number system with its own addition and multiplication. In this way it is like the familiar set $\mathbb{Z}$ which also comes equipped with an addition and a multiplication.

Of course, there is nothing special about the number 5. We can also define $\mathbb{Z}_n$ for any natural number n. Here is the definition:

Definition 17.6. Let $n \in \mathbb{N}$. The equivalence classes of the equivalence relation $\equiv \pmod{n}$ are $[0],[1],[2],\ldots,[n-1]$. The **integers modulo n** is the set $\mathbb{Z}_n = \big\{[0],[1],[2],\ldots,[n-1]\big\}$. Elements of $\mathbb{Z}_n$ can be added by the rule $[a]+[b] = [a+b]$ and multiplied by the rule $[a]\cdot[b] = [ab]$.

Given a natural number n, the set $\mathbb{Z}_n$ is a number system containing n elements. It has many of the algebraic properties that $\mathbb{Z}, \mathbb{R}$ and $\mathbb{Q}$ possess. For example, it is probably obvious to you already that elements of $\mathbb{Z}_n$ obey the commutative laws $[a]+[b] = [b]+[a]$ and $[a]\cdot[b] = [b]\cdot[a]$. You can also verify the distributive law $[a]\cdot([b]+[c]) = [a]\cdot[b]+[a]\cdot[c]$, as follows:

$$\begin{aligned}
[a]\cdot([b]+[c]) &= [a]\cdot[b+c] \\
&= [a(b+c)] \\
&= [ab+ac] \\
&= [ab]+[ac] \\
&= [a]\cdot[b]+[a]\cdot[c].
\end{aligned}$$

The integers modulo n are significant because they more closely fit certain applications than do other number systems such as $\mathbb{Z}$ or $\mathbb{R}$. If you go on to take a course in abstract algebra, then you will work extensively with $\mathbb{Z}_n$ as well as other, more exotic, number systems. (In such a course you will also use all of the proof techniques that we have discussed, as well as the ideas of equivalence relations.)

To close this section we take up an issue that may have bothered you earlier. It has to do with our definitions of addition $[a]+[b] = [a+b]$ and multiplication $[a]\cdot[b] = [ab]$. These definitions define addition and multiplication of equivalence classes in terms of representatives a and b in the equivalence classes. Since there are many different ways to choose such representatives, we may well wonder if addition and multiplication are consistently defined. For example, suppose two people, Alice and Bob, want to multiply the elements [2] and [3] in $\mathbb{Z}_5$. Alice does the calculation as $[2]\cdot[3] = [6] = [1]$, so her final answer is [1]. Bob does it differently. Since $[2] = [7]$ and $[3] = [8]$, he works out $[2]\cdot[3]$ as $[7]\cdot[8] = [56]$. Since $56 \equiv 1 \pmod 5$, Bob's answer is $[56] = [1]$, and that agrees with Alice's answer. Will their answers always agree or did they just get lucky (with the arithmetic)?

The fact is that no matter how they do the multiplication in $\mathbb{Z}_n$, their answers will agree. To see why, suppose Alice and Bob want to multiply the elements $[a], [b] \in \mathbb{Z}_n$, and suppose $[a] = [a']$ and $[b] = [b']$. Alice and Bob do the multiplication as follows:

$$\begin{array}{ll} \text{Alice:} & [a]\cdot[b] = [ab], \\ \text{Bob:} & [a']\cdot[b'] = [a'b']. \end{array}$$

We need to show that their answers agree, that is, we need to show $[ab] = [a'b']$. Since $[a] = [a']$, we know by Theorem 17.1 that $a \equiv a' \pmod n$. Thus $n \mid (a - a')$, so $a - a' = nk$ for some integer k. Likewise, as $[b] = [b']$, we know $b \equiv b' \pmod n$, or $n \mid (b - b')$, so $b - b' = n\ell$ for some integer ℓ. Thus we get $a = a' + nk$ and $b = b' + n\ell$. Therefore:

$$\begin{aligned} ab &= (a' + nk)(b' + n\ell) \\ &= a'b' + a'n\ell + nkb' + n^2k\ell, \\ \text{hence} \quad ab - a'b' &= n(a'\ell + kb' + nk\ell). \end{aligned}$$

This shows $n \mid (ab - a'b')$, so $ab \equiv a'b' \pmod n$, and from that we conclude $[ab] = [a'b']$. Consequently Alice and Bob really do get the same answer, so we can be assured that the definition of multiplication in $\mathbb{Z}_n$ is consistent.

Exercise 8 below asks you to show that addition in $\mathbb{Z}_n$ is similarly consistent.

Exercises for Section 17.5

1. Write the addition and multiplication tables for $\mathbb{Z}_2$.

2. Write the addition and multiplication tables for $\mathbb{Z}_3$.

3. Write the addition and multiplication tables for $\mathbb{Z}_4$.

4. Write the addition and multiplication tables for $\mathbb{Z}_6$.

5. Take $[a], [b] \in \mathbb{Z}_5$. If $[a]\cdot[b] = [0]$, is it necessarily true that either $[a] = [0]$ or $[b] = [0]$?

6. Take $[a], [b] \in \mathbb{Z}_6$. If $[a]\cdot[b] = [0]$, is it necessarily true that either $[a] = [0]$ or $[b] = [0]$?

7. Do the following calculations in $\mathbb{Z}_9$. Exopress your answer as $[a]$ with $0 \le a \le 8$.
(a) $[8] + [8]$ **(b)** $[24] + [11]$ **(c)** $[21]\cdot[15]$ **(d)** $[8]\cdot[8]$

8. Suppose $[a], [b] \in \mathbb{Z}_n$, and $[a] = [a']$ and $[b] = [b']$. Alice adds $[a]$ and $[b]$ as $[a] + [b] = [a + b]$. Bob adds them as $[a'] + [b'] = [a' + b']$. Show that their answers are the same.

17.6 Relations Between Sets

In the beginning of this chapter, we defined a relation on a set A to be a subset $R \subseteq A \times A$. This created a framework that could model any situation in which elements of A are compared to themselves. In this setting, the statement xRy has elements x and y from A on either side of the R because R compares elements from A. But there are other relational symbols that don't work this way. Consider $\in$. The statement $5 \in \mathbb{Z}$ expresses a relationship between 5 and $\mathbb{Z}$ (namely that the element 5 is in the set $\mathbb{Z}$) but 5 and $\mathbb{Z}$ are not in any way naturally regarded as both elements of some set A. To overcome this difficulty, we generalize the idea of a relation on A to a *relation from A to B*.

Definition 17.7. A **relation** from a set A to a set B is a subset $R \subseteq A \times B$. We often abbreviate the statement $(x, y) \in R$ as xRy. The statement $(x, y) \notin R$ is abbreviated as $x \not R y$.

This definition will play a role in our treatment of *functions* in the next chapter.

Example 17.17. Suppose $A = \{1, 2\}$ and $B = \mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Here is a relation from A to B:

$$R = \{(1, \{1\}), (2, \{2\}), (1, \{1, 2\}), (2, \{1, 2\})\} \subseteq A \times B.$$

Note that $1R\{1\}$, $2R\{2\}$, $1R\{1, 2\}$ and $2R\{1, 2\}$. The relation R is the familiar relation $\in$ for the set A, that is, $x\,R\,X$ means exactly the same thing as $x \in X$.

Diagrams for relations from A to B differ from diagrams for relations on A. Since there are two sets A and B in a relation from A to B, we have to draw labeled nodes for each of the two sets. Then we draw arrows from x to y whenever xRy. The following figure illustrates this for Example 17.17.

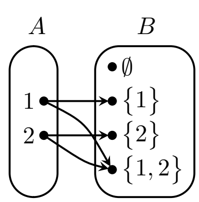


Fig. 17.3 A relation from A to B

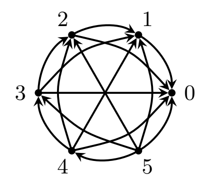
The ideas from this chapter show that any relation (whether it is a familiar one like $\geq$, $\leq$, $=$, $|$, $\in$ or $\subseteq$, or a more exotic one) is really just a set. Therefore the theory of relations is a part of the theory of sets. In the next chapter, we will see that this idea touches on another important mathematical construction, namely functions. We will define a function to be a special kind of relation from one set to another, and in this context we will see that any function is really just a set.

Solutions for Chapter 17

Section 17.1 Exercises

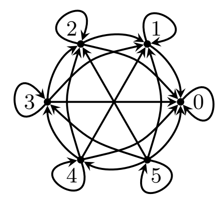
1. Let $A = \{0, 1, 2, 3, 4, 5\}$. Write out the relation R that expresses $>$ on A. Then illustrate it with a diagram.

$$R = \big\{(5,4),(5,3),(5,2),(5,1),(5,0),(4,3),(4,2),(4,1),\\ (4,0),(3,2),(3,1),(3,0),(2,1),(2,0),(1,0)\big\}$$



3. Let $A = \{0, 1, 2, 3, 4, 5\}$. Write out the relation R that expresses $\geq$ on A. Then illustrate it with a diagram.

$$R = \big\{(5,5),(5,4),(5,3),(5,2),(5,1),(5,0),\\ (4,4),(4,3),(4,2),(4,1),(4,0),\\ (3,3),(3,2),(3,1),(3,0),\\ (2,2),(2,1),(2,0),(1,1),(1,0),(0,0)\big\}$$



5. The following diagram represents a relation R on a set A. Write the sets A and R.
Answer: $A = \{0, 1, 2, 3, 4, 5\}$; $R = \{(3,3),(4,3),(4,2),(1,2),(2,5),(5,0)\}$

7. Write the relation $<$ on the set $A = \mathbb{Z}$ as a subset R of $\mathbb{Z} \times \mathbb{Z}$. This is an infinite set, so you will have to use set-builder notation.

Answer: $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in \mathbb{N}\}$

9. How many different relations are there on the set $A = \big\{1, 2, 3, 4, 5, 6\big\}$?

Consider forming a relation $R \subseteq A \times A$ on A. For each ordered pair $(x, y) \in A \times A$, we have two choices: we can either include (x, y) in R or not include it. There are $6 \cdot 6 = 36$ ordered pairs in $A \times A$. By the multiplication principle, there are thus 2^{36} different subsets R and hence also this many relations on A.

11. Answer: $2^{(|A|^2)}$ **13.** Answer: $\neq$ **15.** Answer: $\equiv \pmod 3$

Section 17.2 Exercises

1. Consider the relation $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$ on the set $A = \{a, b, c, d\}$. Which of the properties reflexive, symmetric and transitive does R possess and why? If a property does not hold, say why.

This **is reflexive** because $(x, x) \in R$ (i.e., xRx)for every $x \in A$.
It **is symmetric** because it is impossible to find an $(x, y) \in R$ for which $(y, x) \notin R$.
It **is transitive** because $(xRy \wedge yRz) \Rightarrow xRz$ always holds.

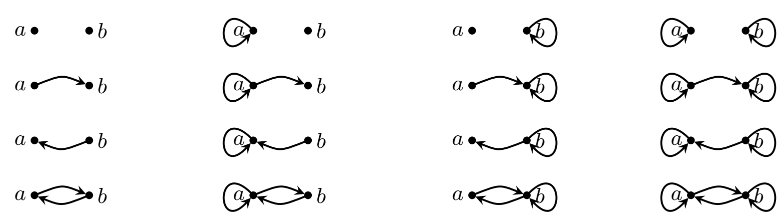
3. Consider the relation $R = \{(a,b),(a,c),(c,b),(b,c)\}$ on the set $A = \{a, b, c\}$. Which of the properties reflexive, symmetric and transitive does R possess and why? If a property does not hold, say why.

This **is not reflexive** because $(a,a) \notin R$ (for example).
It **is not symmetric** because $(a,b) \in R$ but $(b,a) \notin R$.
It **is not transitive** because cRb and bRc are true, but cRc is false.

5. Consider the relation $R = \{(0,0), (\sqrt{2},0), (0,\sqrt{2}), (\sqrt{2},\sqrt{2})\}$ on $\mathbb{R}$. Say whether this relation is reflexive, symmetric and transitive. If a property does not hold, say why.

This **is not reflexive** because $(1,1) \notin R$ (for example).
It **is symmetric** because it is impossible to find an $(x,y) \in R$ for which $(y,x) \notin R$.
It **is transitive** because $(xRy \wedge yRz) \Rightarrow xRz$ always holds.

7. There are 16 possible different relations R on the set $A = \{a,b\}$. Describe all of them. (A picture for each one will suffice, but don't forget to label the nodes.) Which ones are reflexive? Symmetric? Transitive?



Only the four in the right column are reflexive. Only the eight in the first and fourth rows are symmetric. All of them are transitive **except** the first three on the fourth row.

9. Define a relation on $\mathbb{Z}$ by declaring xRy if and only if x and y have the same parity. Say whether this relation is reflexive, symmetric and transitive. If a property does not hold, say why. What familiar relation is this?

This **is reflexive** because xRx since x always has the same parity as x.
It **is symmetric** because if x and y have the same parity, then y and x must have the same parity (that is, $xRy \Rightarrow yRx$).
It **is transitive** because if x and y have the same parity and y and z have the same parity, then x and z must have the same parity. (That is $(xRy \wedge yRz) \Rightarrow xRz$ always holds.)
The relation is congruence modulo 2.

11. Suppose $A = \{a,b,c,d\}$ and $R = \{(a,a),(b,b),(c,c),(d,d)\}$. Say whether this relation is reflexive, symmetric and transitive. If a property does not hold, say why.

This **is reflexive** because $(x,x) \in R$ for every $x \in A$.
It **is symmetric** because it is impossible to find an $(x,y) \in R$ for which $(y,x) \notin R$.
It **is transitive** because $(xRy \wedge yRz) \Rightarrow xRz$ always holds.
(For example $(aRa \wedge aRa) \Rightarrow aRa$ is true, etc.)

13. Consider the relation $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$. Prove that this relation is reflexive and symmetric, and transitive.

Proof. In this relation, xRy means $x - y \in \mathbb{Z}$.

To see that R is reflexive, take any $x \in \mathbb{R}$ and observe that $x - x = 0 \in \mathbb{Z}$, so xRx. Therefore R is reflexive.

To see that R is symmetric, we need to prove $xRy \Rightarrow yRx$ for all $x, y \in \mathbb{R}$. We use direct proof. Suppose xRy. This means $x - y \in \mathbb{Z}$. Then it follows that $-(x - y) = y - x$ is also in $\mathbb{Z}$. But $y - x \in \mathbb{Z}$ means yRx. We've shown xRy implies yRx, so R is symmetric.

To see that R is transitive, we need to prove $(xRy \wedge yRz) \Rightarrow xRz$ is always true. We prove this conditional statement with direct proof. Suppose xRy and yRz. Since xRy, we know $x - y \in \mathbb{Z}$. Since yRz, we know $y - z \in \mathbb{Z}$. Thus $x - y$ and $y - z$ are both integers; by adding these integers we get another integer $(x-y)+(y-z) = x-z$. Thus $x - z \in \mathbb{Z}$, and this means xRz. We've now shown that if xRy and yRz, then xRz. Therefore R is transitive. $\square$

15. Prove or disprove: If a relation is symmetric and transitive, then it is also reflexive.

This is **false**. For a counterexample, consider the relation $R = \{(a, a), (a, b), (b, a), (b, b)\}$ on the set $A = \{a, b, c\}$. This is symmetric and transitive but it is not reflexive.

17. Define a relation $\sim$ on $\mathbb{Z}$ as $x \sim y$ if and only if $|x-y| \leq 1$. Say whether $\sim$ is reflexive, symmetric and transitive.

This is reflexive because $|x - x| = 0 \leq 1$ for all integers x. It is symmetric because $x \sim y$ if and only if $|x - y| \leq 1$, if and only if $|y - x| \leq 1$, if and only if $y \sim x$. It is not transitive because, for example, $0 \sim 1$ and $1 \sim 2$, but is not the case that $0 \sim 2$.

Section 17.3 Exercises

1. Let $A = \{1, 2, 3, 4, 5, 6\}$, and consider the following equivalence relation on A: $R = \big\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (2, 3), (3, 2), (4, 5), (5, 4), (4, 6), (6, 4), (5, 6), (6, 5)\big\}$. List the equivalence classes of R.

The equivalence classes are: $[1] = \{1\}$; $[2] = [3] = \{2, 3\}$; $[4] = [5] = [6] = \{4, 5, 6\}$.

3. Let $A = \{a, b, c, d, e\}$. Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.

Answer: $R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (d, a), (b, c), (c, b)\}$.

5. There are two different equivalence relations on the set $A = \{a, b\}$. Describe them all. Diagrams will suffice.

Answer: $R = \{(a, a), (b, b)\}$ and $R = \{(a, a), (b, b), (a, b), (b, a)\}$

7. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x - 5y$ is even. Prove R is an equivalence relation. Describe its equivalence classes.

To prove that R is an equivalence relation, we must show it's reflexive, symmetric and transitive.

The relation R is reflexive for the following reason. If $x \in \mathbb{Z}$, then $3x - 5x = -2x$ is even. But then since $3x - 5x$ is even, we have xRx. Thus R is reflexive.

To see that R is symmetric, suppose xRy. We must show yRx. Since xRy, we know $3x - 5y$ is even, so $3x - 5y = 2a$ for some integer a. Now reason as follows:

$$\begin{aligned} 3x - 5y &= 2a \\ 3x - 5y + 8y - 8x &= 2a + 8y - 8x \\ 3y - 5x &= 2(a + 4y - 4x). \end{aligned}$$

From this it follows that $3y - 5x$ is even, so yRx. We've now shown xRy implies yRx, so R is symmetric.

To prove that R is transitive, assume that xRy and yRz. (We will show that this implies xRz.) Since xRy and yRz, it follows that $3x-5y$ and $3y-5z$ are both even, so $3x-5y = 2a$ and $3y-5z = 2b$ for some integers a and b. Adding these equations, we get $(3x-5y)+(3y-5z) = 2a+2b$, and this simplifies to $3x-5z = 2(a+b+y)$. Therefore $3x-5z$ is even, so xRz. We've now shown that if xRy and yRz, then xRz, so R is transitive.

We've now shown that R is reflexive, symmetric and transitive, so it is an equivalence relation.

The completes the first part of the problem. Now we move on the second part. To find the equivalence classes, first note that $[0] = \{x \in \mathbb{Z} : xR0\} = \{x \in \mathbb{Z} : 3x - 5 \cdot 0 \text{ is even}\} = \{x \in \mathbb{Z} : 3x \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is even}\}$. Thus the equivalence class [0] consists of all even integers.

Next, note that $[1] = \{x \in \mathbb{Z} : xR1\} = \{x \in \mathbb{Z} : 3x - 5 \cdot 1 \text{ is even}\} = \{x \in \mathbb{Z} : 3x - 5 \text{ is even}\} = \{x \in \mathbb{Z} : x \text{ is odd}\}$. Thus the equivalence class [1] consists of all odd integers.

Consequently there are just two equivalence classes $\{\ldots,-4,-2,0,2,4,\ldots\}$ and $\{\ldots,-3,-1,1,3,5,\ldots\}$.

9. Define a relation R on $\mathbb{Z}$ as xRy if and only if $4 \mid (x+3y)$. Prove R is an equivalence relation. Describe its equivalence classes.

This is reflexive, because for any $x \in \mathbb{Z}$ we have $4 \mid (x+3x)$, so xRx.

To prove that R is symmetric, suppose xRy. Then $4 \mid (x+3y)$, so $x+3y = 4a$ for some integer a. Multiplying by 3, we get $3x+9y = 12a$, which becomes $y+3x = 12a-8y$. Then $y+3x = 4(3a-2y)$, so $4 \mid (y+3x)$, hence yRx. Thus we've shown xRy implies yRx, so R is symmetric.

To prove transitivity, suppose xRy and yRz. Then $4|(x+3y)$ and $4|(y+3z)$, so $x+3y = 4a$ and $y+3z = 4b$ for some integers a and b. Adding these two equations produces $x+4y+3z = 4a+4b$, or $x+3z = 4a+4b-4y = 4(a+b-y)$. Consequently $4|(x+3z)$, so xRz, and R is transitive.

As R is reflexive, symmetric and transitive, it is an equivalence relation.

Now let's compute its equivalence classes.
$[0]=\{x \in \mathbb{Z} : xR0\}=\{x \in \mathbb{Z} : 4 \mid (x+3\cdot 0)\}=\{x \in \mathbb{Z} : 4 \mid x\}=\{\ldots-4,0,4,8,12,16\ldots\}$
$[1]=\{x \in \mathbb{Z} : xR1\}=\{x \in \mathbb{Z} : 4 \mid (x+3\cdot 1)\}=\{x \in \mathbb{Z} : 4 \mid (x+3)\}=\{\ldots-3,1,5,9,13\ldots\}$
$[2]=\{x \in \mathbb{Z} : xR2\}=\{x \in \mathbb{Z} : 4 \mid (x+3\cdot 2)\}=\{x \in \mathbb{Z} : 4 \mid (x+6)\}=\{\ldots-2,2,6,10,\ldots\}$
$[3]=\{x \in \mathbb{Z} : xR3\}=\{x \in \mathbb{Z} : 4 \mid (x+3\cdot 3)\}=\{x \in \mathbb{Z} : 4 \mid (x+9)\}=\{\ldots-1,3,7,11\ldots\}$

11. Prove or disprove: If R is an equivalence relation on an infinite set A, then R has infinitely many equivalence classes.

This is **False**. Counterexample: consider the relation of congruence modulo 2. It is a relation on the infinite set $\mathbb{Z}$, but it has only two equivalence classes.

13. Answer: $m|A|$

15. Answer: 15

Section 17.4 Exercises

1. List all the partitions of the set $A = \{a, b\}$. Compare your answer to the answer to Exercise 5 of Section 17.3.

There are just two partitions $\{\{a\}, \{b\}\}$ and $\{\{a, b\}\}$. These correspond to the two equivalence relations $R_1 = \{(a, a), (b, b)\}$ and $R_2 = \{(a, a), (a, b), (b, a), (b, b)\}$, respectively, on A.

3. Describe the partition of $\mathbb{Z}$ resulting from the equivalence relation $\equiv \pmod 4$.

Answer: The partition is $\{[0], [1], [2], [3]\} =$
$\{\{\ldots, -4, 0, 4, 8, 12, \ldots\}, \{\ldots, -3, 1, 5, 9, 13, \ldots\}, \{\ldots, -2, 2, 4, 6, 10, 14, \ldots\}, \{\ldots, -1, 3, 7, 11, 15, \ldots\}\}$

5. Answer: Congruence modulo 2, or "same parity."

Section 17.5 Exercises

1. Write the addition and multiplication tables for $\mathbb{Z}_2$.

$+$	$[0]$	$[1]$
$[0]$	$[0]$	$[1]$
$[1]$	$[1]$	$[0]$

$\cdot$	$[0]$	$[1]$
$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$

3. Write the addition and multiplication tables for $\mathbb{Z}_4$.

$+$	$[0]$	$[1]$	$[2]$	$[3]$
$[0]$	$[0]$	$[1]$	$[2]$	$[3]$
$[1]$	$[1]$	$[2]$	$[3]$	$[0]$
$[2]$	$[2]$	$[3]$	$[0]$	$[1]$
$[3]$	$[3]$	$[0]$	$[1]$	$[2]$

$\cdot$	$[0]$	$[1]$	$[2]$	$[3]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[2]$	$[3]$
$[2]$	$[0]$	$[2]$	$[0]$	$[2]$
$[3]$	$[0]$	$[3]$	$[2]$	$[1]$

5. Suppose $[a], [b] \in \mathbb{Z}_5$ and $[a] \cdot [b] = [0]$. Is it necessarily true that either $[a] = [0]$ or $[b] = [0]$?

The multiplication table for $\mathbb{Z}_5$ is shown in Section 17.5. In the body of that table, the only place that $[0]$ occurs is in the first row or the first column. That row and column are both headed by $[0]$. It follows that if $[a] \cdot [b] = [0]$, then either $[a]$ or $[b]$ must be $[0]$.

7. Do the following calculations in $\mathbb{Z}_9$, in each case expressing your answer as $[a]$ with $0 \le a \le 8$.

(a) $[8] + [8] = [7]$ **(b)** $[24] + [11] = [8]$ **(c)** $[21] \cdot [15] = [0]$ **(d)** $[8] \cdot [8] = [1]$