QUIZZES AND EXAMS

Week 1 Quiz

Problem 1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x(3 - x)$ and $y = 0$ about the line $x = -1$.

Problem 2. Find the exact length of the curve $x = 1 + 4 \sin(2t)$, $y = 4 \cos(2t) - 3$, $0 \leq t \leq \pi$.

Week 2 Quiz

Problem 1. Find the average value of the function $f(x) = \sin(\pi x)$ on the interval $[-1/2, 1]$.

Problem 2. Find the solution of the differential equation

$$y' = \frac{y}{\sqrt{x-1}}$$

that satisfies the initial condition $y(5) = -3e^4$.

Week 3 Quiz

Problem 1. Determine whether the sequence

$$a_n = \ln \left( \frac{n^2 - 1}{3 + n^2} \right) \text{ for } n \geq 2$$

converges or diverges. If it converges, find the limit.

Problem 2. A sequence $\{a_n\}_n$ is given by

$$a_1 = 1 \text{ and } a_{n+1} = \frac{1}{4}(a_n + 6) \text{ for } n \geq 1.$$

i) Show that $\{a_n\}_n$ is increasing.
ii) Show that $\{a_n\}_n$ is bounded above by 2.
iii) Determine whether the sequence converges or diverges.
Super Quiz 1

Problem 1. (6 points.) Find the value of \( c \) such that the following holds

\[
\sum_{n=2}^{\infty} 2c^n = 1.
\]

Solution. Since the series converges, we have \( |c| < 1 \). We will solve for \( c \), and we will verify this inequality at the end. We have

\[
1 = \sum_{n=2}^{\infty} 2c^n = \sum_{n=0}^{\infty} 2c^n - \sum_{n=0}^{1} 2c^n = \frac{2}{1 - c} - 2 - 2c.
\]

Hence, we have

\[
\frac{2}{1 - c} - 2 - 2c = 1.
\]

Multiplying by \( 1 - c \) and rearranging, we have \( 2c^2 + c - 1 = 0 \). Solving for \( c \), we find two solutions: \( c = -1 \) and \( c = \frac{1}{2} \). Since we must have \( |c| < 1 \), the only valid solution is \( c = \frac{1}{2} \).

\( \square \)

Problem 2. (4 points.) Determine whether the following series converges or diverges; if it converges, find its limit:

\[
\sum_{k=0}^{\infty} \left( \frac{1}{3} \right)^{3-k}.
\]

Solution. The series can be rewritten as follows

\[
\sum_{k=0}^{\infty} \left( \frac{1}{3} \right)^{3-k} = \left( \frac{1}{3} \right)^3 \sum_{k=0}^{\infty} \left( \frac{1}{3} \right)^{-k} = \left( \frac{1}{3} \right)^3 \sum_{k=0}^{\infty} 3^k.
\]

The series on the right-hand side is a geometric series with common ratio equal to \( 3 \geq 1 \), hence the series diverges. \( \square \)

Problem 3. (5 points.) Determine whether the following series converges or diverges; if it converges, find its limit:

\[
\sum_{k=2}^{\infty} \frac{2}{k^2 - 1}.
\]

Solution. This is a telescoping series. Using partial fractions decomposition, one has

\[
\frac{2}{k^2 - 1} = \frac{1}{k - 1} - \frac{1}{k + 1}.
\]
Hence, one has
\[
\sum_{k=2}^{m} \frac{2}{k^2 - 1} = \sum_{k=2}^{m} \left( \frac{1}{k-1} - \frac{1}{k+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{4} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{m-2} - \frac{1}{m} \right) \\
+ \left( \frac{1}{m-1} - \frac{1}{m+1} \right) = 1 + \frac{1}{2} - \frac{1}{m} - \frac{1}{m+1} \xrightarrow{m \to \infty} \frac{3}{2}.
\]
The series thus converges to \(\frac{3}{2}\). \hfill \Box

**Problem 4.** (5 points.) Find the solution of the differential equation
\[(x^2 + 2)y' = xy\]
that satisfies the initial condition \(y(0) = 2\).

**Solution.** Using the method of separation of variables, one has
\[
\int \frac{1}{y} \, dy = \int \frac{x}{x^2 + 2} \, dx.
\]
Hence, one has
\[
\ln |y| = \frac{1}{2} \ln(x^2 + 2) + C
\]
for some constant \(C\). This implies
\[
|y| = e^{\frac{1}{2} \ln(x^2 + 2) + C} = e^{\frac{1}{2} \ln(x^2 + 2)} \cdot e^C,
\]
and finally
\[
y = B \cdot e^{\frac{1}{2} \ln(x^2 + 2)} = B \cdot \left( e^{\ln(x^2 + 2)} \right)^{\frac{1}{2}} = B \sqrt{x^2 + 2}
\]
for some constant \(B\). Using the initial condition, one has \(2 = y(0) = B\sqrt{2}\), hence \(B = \sqrt{2}\).
The solution is thus \(y(x) = \sqrt{2x^2 + 4}\). \hfill \Box

**Midterm 1**

**Problem 1.** (12 points.) A sequence \(\{a_n\}_n\) is given by
\[
a_1 = 7 \quad \text{and} \quad a_{n+1} = \frac{2}{3}(a_n + 2) \quad \text{for} \ n \geq 1.
\]

i) Show that \(\{a_n\}_n\) is decreasing.
ii) Show that \(\{a_n\}_n\) is bounded below by 4.
iii) Determine whether the sequence converges or diverges.
iv) If the sequence converges, find its limit.
Problem 2. (12 points.) Solve the following initial-value problem
\[
\frac{y'}{\cos(x)} = -y \cdot \sin^2(x), \quad y(0) = \pi.
\]

Solution. Using the method of separation of variables, one has
\[
\int \frac{1}{y} \, dy = - \int \sin^2(x) \cos(x) \, dx.
\]
One deduces
\[
\ln |y| = -\frac{\sin^3(x)}{3} + A
\]
for some constant \(A\). Solving for \(y\), one has first
\[
|y| = e^{-\frac{\sin^3(x)}{3} + A} = B \cdot e^{-\frac{\sin^3(x)}{3}}
\]
for some constant \(B\), and then
\[
y = C \cdot e^{-\frac{\sin^3(x)}{3}}
\]
for some constant \(C\). Using the initial condition, one has \(\pi = y(0) = C\), hence the final answer is \(y = \pi \cdot e^{-\frac{\sin^3(x)}{3}}\). \(\square\)

Problem 3. (12 points.) Determine whether the following series converges or diverges; if it converges, find its limit
\[
\sum_{k=1}^{\infty} \frac{6}{k^2 + 2k}.
\]

Solution. Using partial fractions decomposition, one has
\[
\frac{6}{k^2 + 2k} = \frac{3}{k} - \frac{3}{k + 2}.
\]
Hence, the partial sums simplify as follows
\[
\sum_{k=1}^{m} \frac{6}{k^2 + 2k} = \sum_{k=1}^{m} \left( \frac{3}{k} - \frac{3}{k + 2} \right)
\]
\[
= (3 - \frac{3}{3}) + \left( \frac{3}{2} - \frac{3}{4} \right) + \left( 1 - \frac{3}{5} \right) + \cdots + \left( \frac{3}{m-4} - \frac{3}{m+1} \right)
\]
\[
+ \left( \frac{3}{m} - \frac{3}{m+2} \right)
\]
\[
= 3 + \frac{3}{2} - \frac{3}{m+1} - \frac{3}{m+2} \xrightarrow{m \to \infty} 3 + \frac{3}{2} = \frac{9}{2}.
\]
The series thus converges to \(\frac{9}{2}\). \(\square\)
Problem 4. (12 points.) Find the interval of convergence of the following power series
\[ \sum_{k=3}^{\infty} (-1)^{k+1} \frac{2^{2k+1}}{\sqrt{2k+1}} (x-1)^k. \]

Problem 5. (12 points.) Find a power series representation for the following function, and determine the radius of convergence
\[ f(x) = \frac{x^2}{x^3 + 8}. \]

Problem 6. (6 Bonus points.) Find a rational function whose power series representation is equal to the following series
\[ \sum_{n=2}^{\infty} n(n-1)x^n. \]

Week 8 Quiz

Problem 1. Find the area of the parallelogram with vertices \( A = (0,0), B = (1,4), C = (6,6), \) and \( D = (5,2). \)

Problem 2. Find the angle between the vectors \( \langle 1,-1,0 \rangle \) and \( \langle 2,-1,-2 \rangle. \)

Week 9 Super Quiz

Problem 1. Find the Maclaurin series of the following function, and determine its radius of convergence
\[ f(x) = \frac{1}{\sqrt{27 - x^3}}. \]

Solution. Using the binomial series, one has
\[
\begin{align*}
  f(x) &= (27 - x^3)^{-1/3} = 27^{-1/3} \left(1 - \frac{x^3}{27}\right)^{-1/3} \\
  &= \frac{1}{3} \sum_{n=0}^{\infty} \binom{-1/3}{n} \left(-\frac{x^3}{27}\right)^n \\
  &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3}(-\frac{1}{3} - 1) \cdots (-\frac{1}{3} - n + 1) \frac{x^{3n}}{3^{3n+1}}.
\end{align*}
\]

The radius of the binomial series is preserved, hence we have that \(|x^3/27| < 1\), which is equivalent to \(-3 < x < 3\). The radius is 3. \(\square\)
Problem 2. Use series to evaluate the following limit
\[
\lim_{x \to 0} \frac{1 - \cos(2x)}{1 - 3x - e^{-3x}}.
\]

Solution. Since \(x\) is approaching 0, we can replace each function with its Maclaurin series. We obtain
\[
\lim_{x \to 0} \frac{1 - \cos(2x)}{1 - 3x - e^{-3x}} = \lim_{x \to 0} \frac{1 - (1 - \frac{4x^2}{2} + \cdots)}{1 - 3x - (1 - 3x + \frac{9x^2}{2} + \cdots)} = \lim_{x \to 0} \frac{\frac{4x^2}{2}}{\frac{9x^2}{2}} = -\frac{4}{9}.
\]

\[
\square
\]

Problem 3. Find the area of the triangle with vertices \(A = (-1, -1), B = (2, 3),\) and \(C = (5, 1)\).

Solution. We can consider the points \(A, B,\) and \(C\) living on the plan \(z = 0\). The coordinates will then be \(A = (-1, -1, 0), B = (2, 3, 0),\) and \(C = (5, 1, 0)\). The vectors \(\vec{AB}\) and \(\vec{AC}\) are now in \(\mathbb{R}^3\), and we can take their cross product. The area of the triangle is given by
\[
\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} |\langle 3, 4, 0 \rangle \times \langle 6, 2, 0 \rangle| = \frac{|-18k|}{2} = 9.
\]

\[
\square
\]

Problem 4. Determine whether the planes \(x - y + z = 1\) and \(y - 2z = 2\) are parallel, perpendicular, or neither. If they are not parallel, find the line of intersection.

Solution. The normal vector to the plane \(x - y + z = 1\) is \(\vec{n}_1 = \langle 1, -1, 1 \rangle\), and the normal vector to the plane \(y - 2z = 2\) is \(\vec{n}_2 = \langle 0, 1, -2 \rangle\). Since the normal vectors \(\vec{n}_1\) and \(\vec{n}_2\) are not proportional, the planes are not parallel. Since \(\vec{n}_1 \cdot \vec{n}_2 \neq 0\), the two normal vectors are not perpendicular, hence the two planes are not perpendicular. To find the line of intersection, we need a point on the line, and a parallel vector \(\vec{v}\).

A point on the line is on both planes, so we need to find a solution of the system
\[
\begin{align*}
x - y + z &= 1 \\
y - 2z &= 2
\end{align*}
\]

There are infinitely many solutions to this system (corresponding to the line of intersection). To find one point, we can fix one more condition, like \(z = 0\), and solve for \(x\) and \(y\). We find that the point \((3, 2, 0)\) lies on both planes, hence lies on the line of intersection.

To find a parallel vector \(\vec{v}\), we use the fact that \(\vec{v}\) must be orthogonal to both \(\vec{n}_1\) and \(\vec{n}_2\) (since the line must lie on both planes). Hence, we can take
\[
\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 1, 2, 1 \rangle.
\]
The line of intersection has vector equation
\[ \mathbf{r}(t) = \langle 3, 2, 0 \rangle + t\langle 1, 2, 1 \rangle, \]
for \( t \in \mathbb{R} \).

**Week 11 Quiz**

**Problem 1.** Find the length of the curve
\[ \mathbf{r}(t) = \langle 4 \sin(t), 3t, 4 \cos(t) \rangle \]
for \( 0 \leq t \leq \pi \).

**Problem 2.** Find the curvature \( \kappa(t) \) of the curve
\[ \mathbf{r}(t) = \langle 2 - t, 4t^2, 3 + t \rangle. \]
What is the curvature at the point \((2,0,3)\)?

**Midterm 2**

**Problem 1.** (12 points.) Consider the following space curves
\[ \mathbf{r}_1(t) = \langle 1 - t^3, 3, t^2 - 1 \rangle, \]
\[ \mathbf{r}_2(s) = \langle 2 + s, 1 - 2s, 4 + 5s \rangle. \]

i) Find the point of intersection of \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(s) \).

ii) Find the angle of intersection of \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(s) \).

**Problem 2.** (14 points.) i) Find the equation of the plane that contains the point \((1,1,2)\) and the line \( x = 2 + t, y = 1 - t, z = 3t \).

ii) Find the equation of the line passing through the point \((1,1,2)\) and meeting the line
\[ \mathbf{r}(t) = \langle 2, 1, 0 \rangle + t\langle 1, -1, 3 \rangle \]
orthogonally.

**Problem 3.** (12 points.) Consider the curve
\[ \mathbf{r}(t) = \langle \sin(3t), 2t^3, \cos(3t) \rangle. \]

i) Find the length of the curve \( \mathbf{r}(t) \) for \( 0 \leq t \leq 3 \).

ii) Find the point \( P \) on the curve \( \mathbf{r}(t) \) such that the length of the curve \( \mathbf{r}(t) \) between the points \((0,0,1)\) and \( P \) is 52.
Problem 4. (12 points.) Consider the following curve
\[ \vec{r}(t) = (t^3 - 1, t, 1 - t). \]

i) Find the curvature \( \kappa(t) \) of the curve \( \vec{r}(t) \).

ii) Find the point on the curve \( \vec{r}(t) \) where the curvature is 0.

Problem 5. (7 points.) Sketch the graph of the surface
\[ 2x^2 - 8x - y^2 + 2y + 7 - 4z = 0. \]

Problem 6. (7 points.) Compute the following limit
\[ \lim_{(x,y) \to (0,0)} \frac{x^4y^2}{x^4 + y^2}. \]

Week 14 Quiz

Problem 1. Let \( z(x, y) = \cos(xy) \) with \( x = e^s \cdot t \) and \( y = t^3 \). Use the Chain Rule to find the partial derivatives \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \) when \( s = 0 \) and \( t = 1 \).

Problem 2. Let \( f(x, y) = \cos(xy) + xy \).

i) Find the maximum rate of change of \( f(x, y) \) at \( (2, 0) \) and the direction in which it occurs.

ii) Find the directional derivative of \( f(x, y) \) at \( (2, 0) \) in the direction of \( \vec{v} = \langle 1, 1 \rangle \).

Final Exam

Problem 1. (4 points.) Let \( f(y) \) be the force in Newtons of gravity upon a rocket with mass \( m \) kg as a function of the height \( y \) from the Earth’s center:
\[ f(y) = \frac{GmM}{y^2}, \]
where \( M \) is the mass of the earth, and \( G \) is the gravitational constant. Suppose \( GmM = 100 \) N meters-squared.

i) Develop an expression for the work \( W \) in Joules that must be consumed to overcome the gravitational force to get the rocket from the Earth’s surface \( y = 6.3 \times 10^6 \) meters to the low-earth orbit \( y = 8.3 \times 10^6 \) meters.

ii) Compute your result in (i). You may leave an algebraic expression as an answer.

Problem 2. (10 points.) Consider the function \( f(x) = \cos(3x) \) on the interval \([−\pi, \pi]\).

i) Compute the Taylor series of \( f(x) \) centered at the point \( a = 0 \).
ii) Compute the second-order Taylor polynomial approximation $T_2(x)$ of $f(x)$ centered at $a = 0$.

iii) Using Taylor’s inequality, find a bound for the maximum error $|T_2(x) - f(x)|$ between the approximation $T_2(x)$ and $f(x)$ on the interval $[-\pi, \pi]$.

iv) Suppose that for a practical application, we require that the error does not exceed $1/10$ on this interval. Based on your bound in the previous problem, will the second-order Taylor polynomial $T_2(x)$ ensure the required accuracy?

Problem 3. (8 points.) Let $G(x, y) = 4 - x^2 - y^2$. Find the points $(x, y)$ that produce the maximum value of $G$ subject to the constraint $F(x, y) = 2xy = 1$.

Problem 4. (12 points.) Find the interval of convergence of the following power series

$$\sum_{k=2}^{\infty} (-1)^k \frac{2^{2k}}{3k-1} (x-2)^k.$$ 

Problem 5. i) (4 points.) Find the equation of the tangent plane to $z = x^6 + y^6 - 6xy + 4$ at the point $(0, 1, 5)$.

ii) (8 points.) Find the local maximum and minimum values and saddle points of the function

$$f(x, y) = x^6 + y^6 - 6xy + 4.$$ 

Problem 6. i) (7 points.) Find the vector equation of the line of intersection of the planes $x + y - 2z = 2$ and $x - y + 3z = 0$.

ii) (7 points.) Find the equation of the plane containing the line of intersection of the planes $x + y - 2z = 2$ and $x - y + 3z = 0$ and perpendicular to the plane $x - 2y + z = 1$. 