Humboldt Universität zu Berlin
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

December 7, 2009

Exercise Sheet 8

Hand in solutions not later than Monday, December 14.

Exercise 1. Let $M_{m,n}$ be the vector space of $m \times n$-matrices. Let $\mathbb{P}$ be the associated projective space. Let $k \in \{1, \ldots, \min\{m, n\} - 1\}$. Denote with $V_k \subset \mathbb{P}$ the subvariety consisting of points $[M] \in \mathbb{P}$ such that $\operatorname{rank} M \leq k$. Let $\Psi \subset V_k \times G(n-k, n)$ be defined as

$$\Psi := \{(A, \Lambda) \mid A\Lambda = 0\}.$$

Let $\pi_1 : \mathbb{P} \times G(n-k, n) \to \mathbb{P}$ and $\pi_2 : \mathbb{P} \times G(n-k, n) \to G(n-k, n)$ be the natural projections.

i) Show that for every $[\Lambda] \in G(n-k, n)$ we have that $\{A \in M_{m,n} \mid A\Lambda = 0\} = Hom(K^n/\Lambda, K^m)$. Using this, calculate $\dim \pi_2^{-1}(P) \cap \Psi$ for every $P \in G(n-k, n)$. Determine also $\dim \Psi$.
ii) Determine $\dim \pi_1^{-1}(Q) \cap \Psi$. Deduce from this that the codimension of V_k in $\mathbb{P}$ equals $(m-k)(n-k)$.

Exercise 2. (*Blow-up of A_k-singularities*) Let $C_k = V(x^2 + y^{k+1}) \subset \mathbb{A}^2$ be an A_k-singularity. Let $\varphi : Y \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ in $(0, 0)$. Let $\overline{C_k}$ be the strict transform of C_k.

i) Show that $\overline{C_1}$ is isomorphic to two disjoint copies of $\mathbb{A}^1$. Determine $\varphi^{-1}\{(0, 0)\} \cap \overline{C_1}$.
ii) Show that $\overline{C_2}$ is isomorphic to $\mathbb{A}^1$. Determine $\varphi^{-1}\{(0, 0)\} \cap \overline{C_2}$.
iii) For $k > 2$, show that $\overline{C_k}$ is isomorphic to C_{k-2}. Determine $\varphi^{-1}\{(0, 0)\} \cap \overline{C_k}$.

Exercise 3. Let $C_k = V(x^2 + y^{k+1}) \subset \mathbb{A}^2$.

i) How many blows-up are needed, in order to have that the strict transform of C_k is smooth?
ii) Sketch how the exceptional divisor looks like after such a sequence of blow-ups.

Exercise 4. (*Plane Cremona transformations*) Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map defined by $[a_0 : a_1 : a_2] \mapsto [a_1 a_2 : a_0 a_2 : a_0 a_1]$, whenever this makes sense.

i) Show that χ is birational and that $\chi^{-1} = \chi$, i.e., $\chi^2 = id$.
ii) Find open sets $U, V \subset \mathbb{P}^2$ where the restriction of χ is an isomorphism.