Humboldt Universität zu Berlin November 23, 2009
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 6

Hand in solutions not later than Monday, November 30.

Exercise 1. Let $X \subset \mathbb{A}^n$ be an affine variety. Let $\overline{X}$ be its projective closure. Show that the field of rational functions $K(X)$ of X is equal to the field of rational functions $K(\overline{X})$ of $\overline{X}$.

Exercise 2. Let $v_{1,2} : \mathbb{P}^1 \to \mathbb{P}^2$ be the Veronese embedding $(s : t) \mapsto (s^2 : st : t^2)$ and let X be its image. Show that the projective coordinate ring of X and of $\mathbb{P}^1$ are not isomorphic, even though $v_{1,2}$ is an isomorphism.

Exercise 3. Let Y be the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$.

i) Give an equation for Y.

ii) Determine three lines l_1, l_2, l_3 in Y, such that l_1 and l_2 meet in one point, and l_2 and l_3 have empty intersection.

iii) Do l_1 and l_3 meet in one point?

Proposition. *Let M and N be open sets of algebraic varieties, and $\varphi : M \to N$ a morphism whose fibers $\varphi^{-1}(n)$ for $n \in N$ are algebraic varieties, all of the same dimension. Then $\dim(M) = \dim(N) + \dim(\varphi^{-1}(n))$.*

Exercise 4. Let $\mathbb{G}(1, n)$ be the *Grassmann variety* parametrizing lines in $\mathbb{P}^n$. If l is a line in $\mathbb{P}^n$, let $\bar{l}$ denote the point of the Grassmann variety corresponding to l. Let $\mathcal{V} \subset \mathbb{G}(1, n) \times \mathbb{P}^n \times \mathbb{P}^n$ be the set of triples $\{(\bar{l}, p, q) \,|\, p, q \in l \,\text{and}\, p \neq q\}$. Let $\pi_1 : \mathcal{V} \to \mathbb{G}(1, n)$ and $\pi_2 : \mathcal{V} \to (\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta$ be the natural projections.

i) Determine $\dim \pi_2^{-1}(p, q)$, for some $(p, q) \in (\mathbb{P}^n \times \mathbb{P}^n) \setminus \Delta$. Deduce the dimension of $\mathcal{V}$ from this.

ii) Determine $\dim \pi_1^{-1}(\bar{l})$ for some $\bar{l} \in \mathbb{G}(1, n)$. Deduce the dimension of $\mathbb{G}(1, n)$ from this.