Humboldt Universität zu Berlin November 16, 2009
Algebraic Geometry I
Lectures by Prof. Dr. R. N. Kloosterman
Exercises by N. Tarasca

Exercise Sheet 5

Hand in solutions not later than Monday, November 23.

Exercise 1 Let $C_1 = V(y^2 + x^3 + x)$, $C_2 = V(x^3 + y^3 + 1)$ and $C_3 = V(y^3 - 2xy^2 + x^2y + y + 1)$ be three curves in $\mathbb{A}^2$. Determine for $i = 1, 2, 3$ the projective closure $\overline{C_i}$ of C_i and determine the intersection of $\overline{C_i}$ with the line at infinity.

Exercise 2 *[Gathmann's notes, Ex. 3.5.2]* Let $C \subset \mathbb{P}^3$ be the *twisted cubic curve* given by the parametrization

$$\mathbb{P}^1 \to \mathbb{P}^3 \quad (s : t) \mapsto (x : y : z : w) = (s^3 : s^2t : st^2 : t^3).$$

Let $P = (0 : 0 : 1 : 0) \in \mathbb{P}^3$, and let H be the hyperplane defined by $z = 0$. Let φ be the projection from P to H, i.e. the map associating to a point Q of C the intersection point of the unique line through P and Q with H.

i) Show that φ is a morphism.

ii) Determine the equation of the curve $\varphi(C) \subset H \cong \mathbb{P}^2$.

iii) Is $\varphi : C \to \varphi(C)$ an isomorphism onto its image?

Exercise 3 The intersection of two varieties may not be a variety. Let Q_1 and Q_2 be the two quadrics in $\mathbb{P}^3$ defined by $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Prove that $Q_1 \cap Q_2$ is the union of the twisted cubic curve and a line in $\mathbb{P}^3$.

Exercise 4 Let $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal embedding.

i) Give an affine open covering $\{U_1, U_2, U_3, U_4\}$ of $\mathbb{P}^1 \times \mathbb{P}^1$. (Hint: use the standard open covering of $\mathbb{P}^1$.)

ii) Determine $V_i := \Delta^{-1}(U_i) \subset \mathbb{P}^1$.

iii) Show that $\Delta(V_i) = \Delta(\mathbb{P}^1) \cap U_i$ for $i = 1, \dots, 4$.

iv) Show that $\Delta(V_i)$ is closed in U_i.

v) Conclude that the prevariety $\mathbb{P}^1$ is a variety.