**Exercise Sheet 12**

*Hand in solutions not later than Monday, February 1.*

**Exercise 1.** Let \( x, y, z \) be homogeneous coordinates on \( \mathbb{P}^2 \). Determine the Hilbert function of \( V(xy) \subset \mathbb{P}^2 \).

**Exercise 2.** Let \( X \) be a prevariety over an algebraically closed field \( K \), and let \( P \in X \) be a closed point of \( X \). Let \( D \) be defined as \( \text{Spec } K[x]/(x^2) \). Show that \( X(D) \cong T_{X,P} \).

**Exercise 3.** Define \( X \) as \( \text{Spec } K[x, y, z]/(x^2, y^2, xy, xz, yz) \). Show that there exist two subschemes \( S \) and \( T \) of \( X \) such that \( S \cup T = X \), \( \dim T = 0 \) and \( S = X_{\text{red}} \), i.e. \( S \) and \( X \) are the same considered as topological space.

**Exercise 4.** [Gathmann’s notes, Ex. 5.6.13] Let \( X \) be an affine variety, let \( Y \) be a closed subscheme of \( X \) defined by the ideal \( I \subset A(X) \), and let \( \tilde{X} \) be the blow-up of \( X \) at \( I \). Show that:

i) \( \tilde{X} = \text{Proj } (\bigoplus_{d \geq 0} I^d) \), where we set \( I^0 := A(X) \);

ii) The projection map \( \tilde{X} \to X \) is the morphism induced by the ring homomorphism \( I^0 \to \bigoplus_{d \geq 0} I^d \);

iii) The exceptional divisor of the blow-up, i.e. the fiber \( Y \times_X \tilde{X} \) of the blow-up \( \tilde{X} \to X \) over \( Y \), is isomorphic to \( \text{Proj } (\bigoplus_{d \geq 0} I^d/I^{d+1}) \).