Exercise Sheet 10

Hand in solutions not later than Monday, January 11.

Exercise 1. Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ be the rational map defined as $[x : y : z] \mapsto [f_0 : f_1 : f_2 : f_3]$, where

\begin{align*}
    f_0 &= (z + y)(z - y)(y - 2z) \\
    f_1 &= (x + y)(x + 3z)(x + 2y - 3z) \\
    f_2 &= (z - y)(x + 3z)(3x + 2y - z) \\
    f_3 &= (z + y)(x + 2y - 3z)(x - y - 2y).
\end{align*}

i) Check that $f_0, f_1, f_2, f_3$ form a basis for the $K$-vector space of homogeneous degree 3 polynomials in $x, y, z$ that vanishes at $p_1 = [1 : 1 : 1], p_2 = [-1 : 1 : 1], p_3 = [-1 : 1 : 0], p_4 = [-3 : -1 : 1], p_5 = [-1 : 2 : 1], p_6 = [-3 : 2 : 1]$.

ii) Prove that the image of $\varphi$ is contained in the cubic hypersurfaces $X = \{[x : y : z : w] \in \mathbb{P}^3 | zw^2 - z^2w + 8xyw + 2xyz + 6y^2 - 16x^2y = 0\}$.

iii) Prove that $\varphi([1 : 0 : 0])$ and $\varphi([0 : 1 : 0])$ are Eckardt points of $X$.

Exercise 2. Let $X = \{[x : y : z : w] \in \mathbb{P}^3 | zw^2 - z^2w + 8xyw + 2xyz + 6y^2 - 16x^2y = 0\}$ be a cubic hypersurface in $\mathbb{P}^3$ and let $\pi : X \setminus \{p\} \rightarrow \mathbb{P}^2$ be the projection from the point $p := [0 : 0 : 1 : 0] \in X$. Let $U$ be the open subset of $X$ where $w \neq 0$.

i) Show that $\pi|_U : U \rightarrow \mathbb{A}^2$ is surjective, and that for each point $P \in U$ we have that $\pi|_U^{-1}(P)$ consists of either one or two points.

ii) Let $C = \{p \in U | \#\pi|_U^{-1}(p) = 1\}$ be the ramification curve. Show that $C$ is a curve of degree 4.

iii) Consider the line $L = \varphi(V(x + y + 4z) \setminus \{p_1, p_3\})$ on $X$, with notation as in Exercise 1. Show that $\ell = \pi(L)$ is a line in $\mathbb{P}^2$ that intersects the ramification curve $C$ at two distinct points $r_1, r_2$. Show that $\ell$ is tangent to $C$ at both $r_1$ and $r_2$.

Exercise 3. Describe the points of $\text{Spec} \ Z[x]$.

Exercise 4. Let $K$ be an algebraically closed field. Let $f$ be an irreducible polynomial in $K[x, y, z]$ Describe the points of $\text{Spec} \ K[x, y, z]/(f)$. 