

COINVARIANTS OF METAPLECTIC REPRESENTATIONS ON MODULI OF ABELIAN VARIETIES

NICOLA TARASCA

ABSTRACT. We construct spaces of coinvariants at principally polarized abelian varieties with respect to the action of an infinite-dimensional Lie algebra. We show how these spaces globalize to twisted \mathcal{D} -modules on moduli of principally polarized abelian varieties, and we determine the Atiyah algebra of a line bundle acting on them. We prove analogous results on the universal abelian variety. An essential aspect of our arguments involves analyzing the Atiyah algebra of the Hodge and theta line bundles on moduli of abelian varieties and the universal abelian variety.

INTRODUCTION

Spaces of coinvariants have classically been constructed by assigning representations of affine Lie algebras, and more generally, vertex operator algebras, to pointed algebraic curves [TUY, FBZ]. These spaces globalize to quasi-coherent sheaves carrying a twisted \mathcal{D} -module structure on moduli of curves. Said structure is induced by an equivariant action of the Virasoro algebra on vertex operator algebra modules and on moduli of curves with marked points and formal coordinates. The construction generalizes to produce twisted \mathcal{D} -modules on moduli spaces parametrizing pointed curves with extra features: line bundles, principal bundles, and \mathcal{D} -bundles [FS, FBZ, BZN].

Removing curves out of the picture, Arbarello–De Concini [ADC] constructed a moduli space $\widehat{\mathcal{A}}_g$ of suitable extensions of principally polarized abelian varieties which includes, via an extended Torelli map, the moduli space $\widehat{\mathcal{M}}_g$ of algebraic curves with a marked point and a formal coordinate. They showed that an infinite-dimensional symplectic algebra $\mathfrak{sp}(H')$ acts transitively on $\widehat{\mathcal{A}}_g$, extending the transitive action of the Witt algebra on $\widehat{\mathcal{M}}_g$. Here, H' is the vector space of Laurent series in a formal variable with zero constant term and carries a symplectic form induced from the infinite-dimensional Heisenberg algebra. The oscillator representation which injects the Witt algebra into $\mathfrak{sp}(H')$ thus appears as the differential of the extended Torelli map.

The present project started from the realization that the results of [ADC] offer an opening for an extension of the classical conformal field theory from moduli spaces of curves to moduli spaces of abelian varieties. Removing curves out of the construction of coinvariants, we thus build and study twisted \mathcal{D} -modules on moduli spaces of abelian varieties.

MSC2020. 14K10, 17B65, 17B66, 17B69, 14F06 (primary), 14K25, 14L15 (secondary).

Key words and phrases. Infinite-dimensional Lie algebras, moduli of abelian varieties, symplectic and metaplectic algebras, connections and Atiyah algebras, vertex operator algebras, sheaves of coinvariants, conformal field theory.

Our construction proceeds as follows. The symplectic algebra $\mathfrak{sp}(H')$ admits a unique (up to isomorphism) non-trivial central extension, called the *metaplectic algebra* $\mathfrak{mp}(H')$ (this is reviewed in §1.4 and the unicity follows from Proposition 4.1). For a representation V of $\mathfrak{mp}(H')$, we define the space of coinvariants of V at a point $a \in \widehat{\mathcal{A}}_g$ as

$$(0.1) \quad \widehat{\mathbb{V}}(V)_a := V / \mathfrak{sp}_F(H') \cdot V,$$

where $\mathfrak{sp}_F(H')$ is the stabilizer of $\mathfrak{sp}(H')$ at a —see (2.7)—and the action of $\mathfrak{sp}_F(H')$ on V is given by a splitting $\mathfrak{sp}_F(H') \subset \mathfrak{mp}(H')$ —see Proposition 4.3. The space (0.1) is the largest quotient of V on which $\mathfrak{sp}_F(H')$ acts trivially. We show that the spaces (0.1) globalize to a sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carrying a twisted \mathcal{D} -module structure (Theorem 7.1). Furthermore, in case V is an admissible $\mathfrak{mp}(H')$ representation, as per Definition 6.1, we show that the sheaf $\widehat{\mathbb{V}}(V)$ descends on the moduli space \mathcal{A}_g of principally polarized, g -dimensional abelian varieties:

Theorem 1. *For an admissible $\mathfrak{mp}(H')$ representation V , the spaces of coinvariants (0.1) give rise to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on \mathcal{A}_g .*

For instance, the theorem applies to Heisenberg vertex algebras of arbitrary rank and even lattice vertex algebras (Lemma 6.3). The metaplectic algebra $\mathfrak{mp}(H')$ acts on $\widehat{\mathcal{A}}_g$ via the projection $\mathfrak{mp}(H') \rightarrow \mathfrak{sp}(H')$ and replaces the Virasoro algebra of the classical conformal field theory. The action of $\mathfrak{mp}(H')$ on V induces the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$. To explicitly determine it, we use the formalism of Atiyah algebras from Beilinson-Schechtman [BS]. Let Λ be the Hodge line bundle on \mathcal{A}_g , and \mathcal{F}_Λ the Atiyah algebra of Λ , i.e., the sheaf of first-order differential operators acting on Λ . We say that an admissible $\mathfrak{mp}(H')$ representation V is of central charge c if the action of $\mathfrak{mp}(H')$ induces an action of the Virasoro algebra on V of central charge c —see Definition 6.2.

Theorem 2. *For an admissible $\mathfrak{mp}(H')$ representation V of central charge c , the sheaf $\mathbb{V}(V)$ on \mathcal{A}_g carries an action of the Atiyah algebra $\frac{c}{2} \mathcal{F}_\Lambda$. This action induces the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$.*

Moreover, we extend these results over the universal abelian variety. It is shown in [ADC] that $\widehat{\mathcal{A}}_g$ admits a universal family $\widehat{\mathcal{X}}_g \rightarrow \widehat{\mathcal{A}}_g$. This extends the universal family \mathcal{X}_g over the stack \mathcal{A}_g . The space $\widehat{\mathcal{X}}_g$ carries a transitive action of the Lie algebra $\mathfrak{sp}(H') \ltimes H'$. The terms of degree at most two in the Weyl algebra $\widetilde{\mathcal{W}}(H)$ give a non-trivial central extension $\widetilde{\mathcal{W}}_2(H)$ of $\mathfrak{sp}(H') \ltimes H'$ (§1.5). For a representation V of $\widetilde{\mathcal{W}}_2(H)$, we define the space of coinvariants of V at $x \in \widehat{\mathcal{X}}_g$ as

$$(0.2) \quad \widehat{\mathbb{V}}(V)_x := V / (\mathfrak{sp}_F(H') \ltimes F) \cdot V,$$

where $\mathfrak{sp}_F(H') \times F$ is the stabilizer of $\mathfrak{sp}(H') \times H'$ at x . We show that the spaces (0.2) globalize to a twisted \mathcal{D} -module on $\widehat{\mathcal{X}}_g$ (Theorem 7.2). In case V is an admissible $\widetilde{\mathcal{U}}_2(H)$ representation, as per Definition 6.1, we show:

Theorem 3. *For an admissible $\widetilde{\mathcal{U}}_2(H)$ representation V , the spaces of coinvariants (0.2) give rise to a twisted \mathcal{D} -module $\mathbb{V}(V)$ on \mathcal{X}_g .*

The twisted \mathcal{D} -module structure is explicitly determined by the action of an appropriate multiple of the Atiyah algebra \mathcal{F}_Θ of the theta line bundle Θ on \mathcal{X}_g :

Theorem 4. *For an admissible $\widetilde{\mathcal{U}}_2(H)$ representation V of central charge c , the sheaf $\mathbb{V}(V)$ on \mathcal{X}_g carries an action of the Atiyah algebra $-c\mathcal{F}_\Theta$. This action induces the twisted \mathcal{D} -module structure on $\mathbb{V}(V)$.*

To prove the above statements, we need to show some auxiliary results, which are of independent interest. It is shown in [ADCKP] that for $g \geq 3$, there are canonical isomorphisms

$$\begin{aligned} H^2(\text{Witt}, \mathbb{C}) &\cong H^2(\mathcal{M}_g, \mathbb{C}), \\ H^2(\text{Witt} \times H', \mathbb{C}) &\cong H^2(\mathcal{P}'_d, \mathbb{C}), \end{aligned}$$

where \mathcal{M}_g is the moduli space of smooth genus g curves, and \mathcal{P}'_d is the moduli space of quadruples (C, P, v, \mathcal{L}) , where C is a smooth genus g curve, P is a point of C , v is a nonzero tangent vector to C at P , and \mathcal{L} is a degree d line bundle on C . We show that these isomorphisms extend to give:

Theorem 5. *For $g \geq 3$, one has canonical isomorphisms*

$$\begin{aligned} H^2(\mathfrak{sp}(H'), \mathbb{C}) &\cong H^2(\mathcal{A}_g, \mathbb{C}), \\ H^2(\mathfrak{sp}(H') \times H', \mathbb{C}) &\cong H^2(\mathcal{X}_g, \mathbb{C}). \end{aligned}$$

As an immediate consequence, we obtain:

Theorem 6. *(i) For $g \geq 3$, the line bundle Λ on $\widehat{\mathcal{A}}_g$ carries an action of $\mathfrak{mp}(H')$ by first-order differential operators extending the transitive action of $\mathfrak{sp}(H')$ on $\widehat{\mathcal{A}}_g$ and with the central element $\mathbf{1} \in \mathfrak{mp}(H')$ acting as multiplication by 2 on the fibers of $\Lambda \rightarrow \widehat{\mathcal{A}}_g$.*
(ii) For $g \geq 3$, the line bundle Θ on $\widehat{\mathcal{X}}_g$ carries an action of $\widetilde{\mathcal{U}}_2(H)$ by first-order differential operators extending the transitive action of $\mathfrak{sp}(H') \times H'$ on $\widehat{\mathcal{X}}_g$ and with the central element $\mathbf{1} \in \widetilde{\mathcal{U}}_2(H)$ acting as multiplication by -1 on the fibers of $\Theta \rightarrow \widehat{\mathcal{X}}_g$.

Moreover, we study group (ind)-schemes of symplectic automorphisms in §3, generalizing the group (ind)-schemes of ring automorphisms from [BD, FBZ]. Our constructions of the sheaves of coinvariants on \mathcal{A}_g and \mathcal{X}_g are thus variations on the formalism of localization of modules over Harish-Chandra pairs (§9.1).

The present results are intended to initiate the study of coinvariants on abelian varieties and their moduli. We mention some natural directions in §9 and will return to these in the near future.

The paper is structured as follows. We review the algebraic and geometric background in §§1 and 2, respectively. In §3 we define and study a group scheme which acts transitively on the fibers of the natural map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$, together with related group (ind-)schemes. In §4 we start the study of the cohomology of the Lie algebras $\mathfrak{sp}(H')$ and $\mathfrak{sp}(H') \ltimes H'$. In §5 we prove Theorem 5.1, which establishes two canonical isomorphisms of H^2 spaces, and show how this implies Theorems 5 and 6. Next, an intermezzo in §6 contains some required properties of metaplectic representations. We construct sheaves of coinvariants on $\widehat{\mathcal{A}}_g$ and $\widehat{\mathcal{X}}_g$ in §7 and descend them on \mathcal{A}_g and \mathcal{X}_g in §8, where we prove Theorems 1–4. We conclude with some final remarks in §9. Throughout, we work over \mathbb{C} .

1. HEISENBERG, METAPLECTIC, AND VIRASORO ALGEBRAS

We review here the infinite-dimensional Lie algebras from [ADCK, ADC] which will play a role in the paper. The complex Lie algebras from these references are here promoted to functors on commutative \mathbb{C} -algebras, as needed for our geometric constructions. Moreover, emphasis is given to the two-cocycles defining the relevant central extensions, later needed in §§4–5.

Here is some notation used throughout. For a topological vector space V , a subset $B \subset V$ is said to *topologically generate* V if finite linear combinations of elements of B form a dense open subspace of V . Also, let $\mathfrak{gl}(V)$ be the Lie algebra of continuous linear endomorphisms of V .

1.1. The algebra of formal Laurent series. Let H be the functor which assigns to a \mathbb{C} -algebra R the R -algebra $H(R) := R((t))$ of formal Laurent series in a parameter t . One has a decomposition

$$H = H_+ \oplus H_-, \quad \text{where } H_+(R) := R[[t]] \quad \text{and} \quad H_-(R) := t^{-1}R[t^{-1}].$$

Also, let H' be the functor which assigns to R the R -module $H'(R) \subset H(R)$ of Laurent series with zero constant term:

$$H' := H'_+ \oplus H_- \quad \text{where } H'_+ := tH_+.$$

The R -module $H(R)$ is endowed with the t -adic topology. This is the topology that has the collection of cosets

$$\{f + t^N H_+ \mid f \in H \text{ and } N \geq 1\}$$

as a basis of open subsets. This gives H the structure of a complete topological space, topologically generated over R by $b_i := t^i$ for $i \in \mathbb{Z}$.

We will repeatedly use the following result from [ADCK]: the Lie algebra $\mathfrak{gl}(H)$ has a canonical two-cocycle given by

$$(1.1) \quad \psi(A, B) := \text{Tr}(\pi_+ A \pi_- B \pi_+ - \pi_+ B \pi_- A \pi_+) \quad \text{for } A, B \in \mathfrak{gl}(H),$$

where $\pi_+ : H \rightarrow H_+$ and $\pi_- : H \rightarrow H_-$ are the natural projections.

1.2. The Heisenberg algebra. Consider the symplectic form $\langle \cdot, \cdot \rangle$ on H given by

$$(1.2) \quad \langle f, g \rangle := -\operatorname{Res}_{t=0} f dg \quad \text{for } f, g \in H.$$

This only depends on df and dg , for $f, g \in H$, and restricts to a nondegenerate symplectic form on H' . The *Heisenberg algebra* is the Lie algebra structure on H given by the Lie bracket $\langle \cdot, \cdot \rangle$. As (1.2) is continuous with respect to the t -adic topology, the Heisenberg algebra H is a complete topological Lie algebra. Moreover, (1.2) is invariant by changes of the parameter t . Note that $\langle f, g \rangle = \psi(f, g)$, where ψ is from (1.1), and $f, g \in H$ act on H by multiplication. Thus, the Heisenberg algebra H is the central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow H \rightarrow H' \rightarrow 0$$

given by the restriction of the two-cocycle ψ , where $\mathbf{1} := b_0$, and \mathfrak{gl}_1 is the functor assigning to a \mathbb{C} -algebra R the trivial Lie algebra R .

1.3. The enveloping algebra $\mathcal{U}(H)$. Let $\mathcal{U}(H)$ be the quotient of the universal enveloping algebra of the Heisenberg algebra H by the two-sided ideal generated by $\mathbf{1} - 1$, where $\mathbf{1}$ is the central element of H , and 1 is the unit in the universal enveloping algebra. The algebra $\mathcal{U}(H)$ has a canonical filtration $\mathcal{U}_0(H) \subset \mathcal{U}_1(H) \subset \dots$, where $\mathcal{U}_0(H) = \mathfrak{gl}_1 \mathbf{1}$, and $\mathcal{U}_N(H)$ for $N \in \mathbb{N}$ assigns to a \mathbb{C} -algebra R the R -module generated by the products $f_1 \cdots f_m$ with $m \leq N$ and $f_i \in H$ for $i \leq N$.

Remarkably, $\mathcal{U}_2(H)$ is a Lie subalgebra of $\mathcal{U}(H)$ for the Lie bracket given by the commutator. Explicitly, the Lie bracket in $\mathcal{U}_2(H)$ is given by

$$(1.3) \quad \begin{aligned} [f, g] &= \langle f, g \rangle, \\ [f, \mathbf{1}] &= [fg, \mathbf{1}] = 0, \\ [fg, k] &= \langle f, k \rangle g + \langle g, k \rangle f, \\ [fg, kl] &= \langle g, k \rangle fl + \langle f, k \rangle gl + \langle g, l \rangle kf + \langle f, l \rangle kg \end{aligned}$$

for $f, g, k, l \in H'$. One has

$$\mathcal{U}_2(H)/\mathfrak{gl}_1 \mathbf{1} \cong S^2(H') \ltimes H'$$

as Lie algebras, where $S^2(H')$ is the second symmetric power of H' , and the semidirect product is given by the action of $S^2(H')$ on H' induced by

$$(1.4) \quad fg : H' \rightarrow H', \quad k \mapsto \langle f, k \rangle g + \langle g, k \rangle f, \quad \text{for } f, g, k \in H'.$$

Hence, $\mathcal{U}_2(H)$ is a central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \mathcal{U}_2(H) \rightarrow S^2(H') \ltimes H' \rightarrow 0.$$

To identify the corresponding two-cocycle, arguing by cases, one shows that

$$\begin{aligned} [: fg :, : kl :] &= -\frac{1}{2} \psi(fg, kl) \mathbf{1} \\ &\quad + \langle g, k \rangle : fl : + \langle f, k \rangle : gl : + \langle g, l \rangle : kf : + \langle f, l \rangle : kg : \end{aligned}$$

in $\mathcal{U}_2(H)$ is compatible with (1.3), where $:$ denotes the normal order product giving a specific lift of elements from $S^2(H')$ to $\mathcal{U}_2(H)$, and ψ is the restriction of the two-cocycle in (1.1). Hence, the two-cocycle c defining $\mathcal{U}_2(H)$ is given by

$$c(fg, kl) = -\frac{1}{2} \psi(fg, kl), \quad c(f, g) = \langle f, g \rangle, \quad c(fg, k) = 0$$

for $f, g, k, l \in H'$.

We will need a completion of $\mathcal{U}_2(H)$. For this, we first introduce the completion of $S^2(H')$ given by the symplectic algebra and its corresponding extension.

1.4. Symplectic and metaplectic algebras. The *symplectic algebra* $\mathfrak{sp}(H')$ is the Lie subalgebra of $\mathfrak{gl}(H')$ defined as

$$\mathfrak{sp}(H') := \{X \in \mathfrak{gl}(H') \mid \langle Xa, b \rangle + \langle a, Xb \rangle = 0, \text{ for all } a, b \in H'\}.$$

The action of $S^2(H')$ on H' induced by (1.4) realizes $S^2(H')$ as a dense Lie subalgebra of $\mathfrak{sp}(H')$.

The *metaplectic algebra* $\mathfrak{mp}(H')$ is the central extension

$$(1.5) \quad 0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \mathfrak{mp}(H') \rightarrow \mathfrak{sp}(H') \rightarrow 0$$

defined by the two-cocycle

$$(1.6) \quad -\frac{1}{2} \psi(X, Y) \quad \text{for } X, Y \in \mathfrak{sp}(H')$$

where ψ is the restriction of (1.1).

1.5. The Weyl algebra. The *Weyl algebra* $\widetilde{\mathcal{U}}(H)$ is the completion of the algebra $\mathcal{U}(H)$ with respect to the topology in which the basis of open neighborhoods of 0 is formed by the left ideals of the submodules

$$t^N H_+ \subset H \subset \widetilde{\mathcal{U}}(H) \quad \text{for } N \in \mathbb{Z}$$

[FBZ, §2.1.2]. Elements of $\widetilde{\mathcal{U}}(H)$ can be described as possibly infinite series of the form $A_0 + \sum_{i \geq 1} A_i b_i$ with A_i equal to a finite linear combination of finite formal products of elements in $\{b_j\}_{j \in \mathbb{Z}}$ for $i \geq 0$.

The closure $\widetilde{\mathcal{U}}_2(H)$ of $\mathcal{U}_2(H)$ in $\widetilde{\mathcal{U}}(H)$ is a complete topological Lie subalgebra of $\widetilde{\mathcal{U}}(H)$ for the Lie bracket given by the commutator. Explicitly, consider first the semidirect product $\mathfrak{sp}(H') \ltimes H'$ given by the action

$$[X, f] = X(f) \quad \text{for } X \in \mathfrak{sp}(H') \text{ and } f \in H'.$$

Hence, the Lie bracket for $\mathfrak{sp}(H') \ltimes H'$ is

$$[X + f, Y + g] = XY - YX + X(g) - Y(f)$$

for $X, Y \in \mathfrak{sp}(H')$ and $f, g \in H'$. Then, one has a central extension

$$(1.7) \quad 0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \widetilde{\mathcal{U}}_2(H) \rightarrow \mathfrak{sp}(H') \ltimes H' \rightarrow 0$$

with two-cocycle c given by

$$(1.8) \quad c(X, Y) = -\frac{1}{2} \psi(X, Y), \quad c(f, g) = \langle f, g \rangle, \quad c(X, f) = 0$$

for $X, Y \in \mathfrak{sp}(H')$ and $f, g \in H'$, where ψ is the restriction of (1.1). This is compatible with the Lie bracket for $\mathcal{W}_2(H)$ in (1.3).

The metaplectic algebra $\mathfrak{mp}(H')$ is realized as a Lie subalgebra of $\widetilde{\mathcal{W}_2}(H)$ via the injection

$$(1.9) \quad \mathfrak{mp}(H') \hookrightarrow \widetilde{\mathcal{W}_2}(H), \quad \mathbf{1} \mapsto \mathbf{1}, \quad X \mapsto X \quad \text{for } X \in \mathfrak{sp}(H').$$

1.6. Witt and Virasoro algebras. The *Witt algebra* Witt is the functor assigning to a \mathbb{C} -algebra R the Lie algebra $R((t))\partial_t$ of continuous derivations of $R((t))$. It is topologically generated over R by $L^p := -t^{p+1}\partial_t$ for $p \in \mathbb{Z}$, with relations $[L_p, L_q] = (p - q)L_{p+q}$ for $p, q \in \mathbb{Z}$. The oscillator representation of Witt gives an injection of Lie algebras

$$(1.10) \quad \tau: \text{Witt} \hookrightarrow \mathfrak{sp}(H'), \quad L_p \mapsto \frac{1}{2} \sum_i b_{-i} b_{i+p}$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$.

The *Virasoro algebra* Vir is the central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \text{Vir} \rightarrow \text{Witt} \rightarrow 0$$

with Lie bracket given by

$$[L_p, L_q] = (p - q)L_{p+q} + \frac{1}{12}(p^3 - p)\delta_{p+q,0} \mathbf{1},$$

where $\delta_{p+q,0} = 1$ when $p+q=0$, and $\delta_{p+q,0} = 0$ otherwise. The two-cocycle defining this extension is the restriction over Witt of the two-cocycle (1.6). The map τ is lifted by the injection

$$(1.11) \quad \hat{\tau}: \text{Vir} \hookrightarrow \mathfrak{mp}(H'), \quad L_p \mapsto \frac{1}{2} \sum_i : b_{-i} b_{i+p} :, \quad \mathbf{1} \mapsto \mathbf{1}$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$.

1.7. The map σ and the Lie algebra \mathfrak{D} . We will need the Lie algebra injection

$$(1.12) \quad \sigma: \text{Witt} \times H' \hookrightarrow \mathfrak{sp}(H') \times H', \quad L_p \mapsto \frac{1}{2} \sum_i b_{-i} b_{i+p} - \frac{p+1}{2} b_p, \quad b_q \mapsto b_q$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$. For $p=0$, one simply has $L_0 \mapsto \frac{1}{2} \sum_i b_{-i} b_i$, since b_0 vanishes in H' .

Remark 1.1. There are infinitely many ways of injecting $\text{Witt} \times H'$ in $\mathfrak{sp}(H') \times H'$: by replacing in (1.12) the coefficient $-\frac{p+1}{2}$ with $a(p+1)$ for $a \in \mathbb{C}$, one obtains a one-parameter family of such injections, as in the Segal-Sugawara construction [FBZ, 5.2.8]. The case $a = -\frac{1}{2}$ is specifically needed in §2.6.

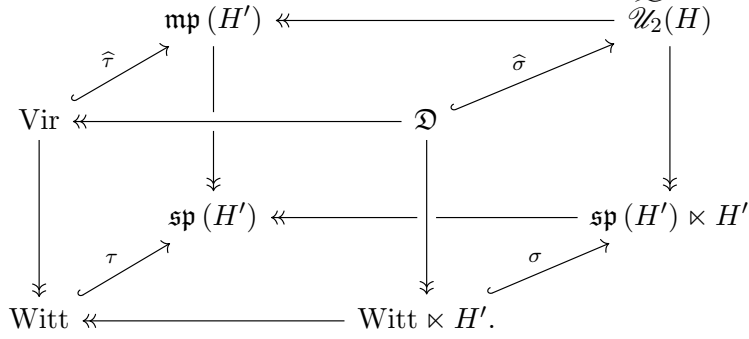


FIGURE 1. The main Lie algebra maps.

Finally, let \mathfrak{D} be the central extension

$$0 \rightarrow \mathfrak{gl}_1 \mathbf{1} \rightarrow \mathfrak{D} \rightarrow \text{Witt} \times H' \rightarrow 0$$

defined by the restriction to $\text{Witt} \times H'$ of the two-cocycle ψ from (1.1) such that Witt acts on H as in (1.10) and H' acts on H by multiplication. The map σ is lifted by the injection

(1.13)

$$\widehat{\sigma}: \mathfrak{D} \hookrightarrow \widetilde{\mathcal{U}}_2(H), \quad L_p \mapsto \frac{1}{2} \sum_i : b_{-i} b_{i+p} : - \frac{p+1}{2} b_p, \quad b_q \mapsto b_q, \quad \mathbf{1} \mapsto \mathbf{1}$$

where the sum is over $i \in \mathbb{Z} \setminus \{0, -p\}$.

The main Lie algebra maps of this section are summarized by the commutative diagram in Figure 1. There, the four horizontal surjections admit Lie algebra splittings, and the left and right squares are Cartesian.

2. EXTENDED ABELIAN VARIETIES AND THEIR MODULI

First, we review some geometric constructions and results from [ADC] related to suitable extensions of abelian varieties and their moduli. Theorems 2.2 and 2.3 play a key role in later sections. Then, we conclude with a review of the degree-2 cohomology classes on the moduli spaces in consideration.

2.1. Extended abelian varieties. Recall the symplectic form \langle, \rangle on H' from (1.2). An *extended principally polarized abelian variety* (extended PPAV, for short) of dimension g is a triple (Z, F, L) , where

Z is a Lagrangian subspace of H' ,

F is a codimension g subspace of Z , and

L is a rank $2g$ lattice in F^\perp/F ,

satisfying four suitable conditions. The form \langle, \rangle on H' induces a nondegenerate symplectic form on F^\perp/F , which will still be denoted as \langle, \rangle . The

first three conditions imposed on the triple (Z, F, L) are:

$$(2.1) \quad Z \cap H'_+ = 0,$$

$$(2.2) \quad L_{\mathbb{R}} \cap A = 0 \quad \text{where} \quad L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad A := F^{\perp} \cap H'_+,$$

$$(2.3) \quad \frac{1}{2\pi i} \langle \cdot, \cdot \rangle \text{ is unimodular on } L.$$

Condition (2.1) implies the decompositions

$$H' = Z \oplus H'_+ \quad \text{and} \quad F^{\perp}/F = Z/F \oplus A$$

into maximal isotropic subspaces. It follows that F^{\perp}/F has dimension $2g$. Condition (2.2) implies that: the projection $F^{\perp}/F \rightarrow Z/F$ induces a real isomorphism $L_{\mathbb{R}} \cong Z/F$; the real structure on F^{\perp}/F induced by $L_{\mathbb{R}}$ identifies A with the conjugate of Z/F ; and the form on Z/F given by

$$B(u, v) := \frac{1}{\pi} \langle \bar{u}, v \rangle \quad \text{for } u, v \in Z/F$$

is Hermitian. The fourth and last condition on the triple (Z, F, L) is

$$(2.4) \quad \text{the form } B \text{ on } Z/F \text{ is positive definite.}$$

A triple (Z, F, L) satisfying the four conditions (2.1)-(2.4) determines an abelian variety X of dimension g , which is the quotient of the g -dimensional space Z/F by the image of L under the projection $F^{\perp}/F \rightarrow Z/F$, equipped with the polarization B , which is principal by (2.3). More precisely, the triple (Z, F, L) is equivalent to an isomorphism class of extensions

$$(2.5) \quad 0 \rightarrow H'_+ \rightarrow H'/K \rightarrow X \rightarrow 0$$

where K is the preimage of L under the projection $F^{\perp} \rightarrow F^{\perp}/F$.

2.2. Moduli of extended PPAVs. In [ADC], the set $\widehat{\mathcal{A}}_g$ of extended PPAVs of dimension g is endowed with the structure of an infinite-dimensional analytic manifold. We sketch here the construction. Consider the infinite-dimensional analytic manifold

$$\widehat{\mathcal{H}}_g := \widetilde{\mathcal{S}}^2(H'_+) \times \mathcal{B}_g(H'_+) \times \mathcal{H}_g$$

where

$$\widetilde{\mathcal{S}}^2(H'_+) := \left\{ \varphi: H_- \rightarrow H'_+ \left| \begin{array}{l} \varphi \text{ is linear and symmetric,} \\ \text{i.e., } \langle a, \varphi(b) \rangle = \langle \varphi(a), b \rangle \text{ for all } a, b \in H_- \end{array} \right. \right\},$$

$$\mathcal{B}_g(H'_+) := \text{the manifold of frames of } (H'_+)^g,$$

$$\mathcal{H}_g := \text{the Siegel upper half-space of degree } g.$$

The symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ acts freely and properly discontinuously on $\widehat{\mathcal{H}}_g$, hence the quotient $\widehat{\mathcal{H}}_g / \mathrm{Sp}(2g, \mathbb{Z})$ has a natural structure of an infinite-dimensional analytic manifold. Moreover, one has a bijection of sets

$$(2.6) \quad \widehat{\mathcal{H}}_g / \mathrm{Sp}(2g, \mathbb{Z}) \rightarrow \widehat{\mathcal{A}}_g.$$

Hence, (2.6) induces on $\widehat{\mathcal{A}}_g$ the structure of an infinite-dimensional analytic manifold [ADC, §3]. One has a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{H}}_g & \xrightarrow{/\mathrm{Sp}(2g,\mathbb{Z})} & \widehat{\mathcal{A}}_g \\ \downarrow & & \downarrow \\ \mathcal{H}_g & \xrightarrow{/\mathrm{Sp}(2g,\mathbb{Z})} & \mathcal{A}_g \end{array}$$

where \mathcal{A}_g is the moduli space of principally polarized abelian varieties of dimension g . The action of $\mathrm{Sp}(2g, \mathbb{Z})$ on \mathcal{H}_g has finite nontrivial stabilizers, hence at least an orbifold structure is required on \mathcal{A}_g . Here we consider \mathcal{A}_g as the resulting quotient stack. This is a separated, smooth Deligne-Mumford stack. From [ADC], $\widehat{\mathcal{A}}_g$ is rational homotopy equivalent to \mathcal{A}_g .

Remark 2.1. To treat $\widehat{\mathcal{A}}_g$ and \mathcal{A}_g on an equal footing, we will consider families $(Z, F, L) \rightarrow S$ of extended PPAVs of dimension g over a smooth scheme S . We will construct sheaves of coinvariants on such families in §7 and descend them to families of PPAVs in §8, thus yielding sheaves on \mathcal{A}_g .

2.3. Symplectic uniformization of moduli of extended PPAVs. An action of a Lie algebra \mathfrak{g} on a variety X over \mathbb{C} is a homomorphism of Lie algebras $\mathfrak{g} \rightarrow \mathrm{Vect}(X)$, where $\mathrm{Vect}(X)$ is the Lie algebra of regular vector fields on X . The action is said to be *transitive* if for every $x \in X$, the evaluation map $\mathfrak{g} \rightarrow T_x(X)$ is surjective.

Theorem 2.2 (Uniformization of $\widehat{\mathcal{A}}_g$ [ADC]). *The moduli space $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}(H')$. For (Z, F, L) in $\widehat{\mathcal{A}}_g$, one has*

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \mathfrak{sp}_F(H') \setminus \mathfrak{sp}(H')$$

where

$$(2.7) \quad \mathfrak{sp}_F(H') := \left\{ X \in \mathfrak{sp}(H') \mid X(F^\perp) \subseteq F \right\}.$$

We sketch the proof from [ADC]: using the map (2.6), and since the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\widehat{\mathcal{H}}_g$ is properly discontinuous, one deduces

$$T_{(Z,F,L)}(\widehat{\mathcal{A}}_g) \cong \widetilde{S}^2(H'_+) \times (H'_+ \otimes Z/F) \times S^2(Z/F).$$

This is then shown to be isomorphic to $\mathfrak{sp}_F(H') \setminus \mathfrak{sp}(H')$. We emphasize that the subspace

$$(2.8) \quad \widetilde{S}^2(H'_+) \times (H'_+ \otimes Z/F)$$

is the tangent space to the fiber of the forgetful map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ at (Z, F, L) .

2.4. The universal extended PPAV. The moduli space $\widehat{\mathcal{A}}_g$ admits a universal family $\widehat{\mathcal{X}}_g \rightarrow \widehat{\mathcal{A}}_g$. The fiber over a point (Z, F, L) in $\widehat{\mathcal{A}}_g$ is

$$(H'/K) \times_G \mathcal{Q}$$

where G is the group of points of order 2 in H'/K , and \mathcal{Q} is the G -torsor of integral quadratic forms q on L satisfying

$$q(a) + q(b) - q(a+b) \equiv \frac{1}{2\pi i} \langle a, b \rangle \pmod{2}.$$

The need for the twist by G becomes clear when one defines the maps in (2.9). Points of $\widehat{\mathcal{X}}_g$ are denoted as (Z, F, L, \bar{h}, q) . The map $\widehat{\mathcal{X}}_g \rightarrow \widehat{\mathcal{A}}_g$ has no zero-section, however the fiber over (Z, F, L) in $\widehat{\mathcal{A}}_g$ is non-canonically isomorphic to H'/K . From [ADC], $\widehat{\mathcal{X}}_g$ is rational homotopy equivalent to the universal family \mathcal{X}_g over the stack \mathcal{A}_g . More generally, we will consider families $(Z, F, L, \bar{h}, q) \rightarrow S$ over a smooth scheme S .

2.5. Uniformization of the universal extended PPAV. Recall the Lie algebra $\mathfrak{sp}(H') \ltimes H'$ from §1.5.

Theorem 2.3 (Uniformization of $\widehat{\mathcal{X}}_g$ [ADC]). *The moduli space $\widehat{\mathcal{X}}_g$ carries a transitive action of $\mathfrak{sp}(H') \ltimes H'$. For (Z, F, L, \bar{h}, q) in $\widehat{\mathcal{X}}_g$, one has*

$$T_{(Z, F, L, \bar{h}, q)}(\widehat{\mathcal{X}}_g) \cong \mathfrak{sp}_F(H') \ltimes F \setminus \mathfrak{sp}(H') \ltimes H'.$$

2.6. The case of curves and line bundles. The moduli spaces $\widehat{\mathcal{A}}_g$ and $\widehat{\mathcal{X}}_g$ are naturally extensions of moduli spaces of curves and line bundles. To see this, let $\widehat{\mathcal{M}}_g$ be the moduli space of triples (C, P, t) , where C is a smooth genus g curve, P is a point of C , and t is a formal coordinate at P [ADCKP]. From [ADC], there is an *extended Torelli map* $\widehat{\mathcal{M}}_g \rightarrow \widehat{\mathcal{A}}_g$ mapping (C, P, t) to a triple (Z, F, L) with identifications

$$F \cong H^0(C \setminus P, \mathcal{O}_C), \quad Z/F \cong H^1(C, \mathcal{O}_C), \quad L \cong H^1(C, \mathbb{Z}).$$

The isotropicity of F with respect to (1.2) follows from the residue theorem.

Moreover, let $\widehat{\mathcal{P}}_d \rightarrow \widehat{\mathcal{M}}_g$ be the *relative Picard variety* parametrizing isomorphism classes of quintuples $(C, P, t, \mathcal{L}, \varphi)$ where \mathcal{L} is a line bundle of degree d on C and φ is a trivialization of the restriction of \mathcal{L} to the formal disc around P . For $d = g - 1$, one has a commutative diagram

$$(2.9) \quad \begin{array}{ccc} \widehat{\mathcal{P}}_{g-1} & \longrightarrow & \widehat{\mathcal{X}}_g \\ \downarrow & & \downarrow \\ \widehat{\mathcal{M}}_g & \longrightarrow & \widehat{\mathcal{A}}_g \end{array}$$

where $(C, P, t, \mathcal{L}, \varphi)$ in $\widehat{\mathcal{P}}_{g-1}$ is mapped to (Z, F, L, \bar{h}, q) in $\widehat{\mathcal{X}}_g$ such that q is arbitrary in \mathcal{Q} , and \bar{h} is the unique point of H'/K given by the isomorphism class of $(\mathcal{L} \eta^{-1}, \varphi dt^{-\frac{1}{2}})$, where η is the unique theta-characteristic on C

identified by q via Riemann's theorem [ADC, (3.23)]. By the construction of $\widehat{\mathcal{X}}_g$, this map is independent of the choice of q .

Recall the commutative diagram

$$(2.10) \quad \begin{array}{ccc} \text{Witt} \times H' & \xrightarrow{\sigma} & \mathfrak{sp}(H') \times H' \\ \downarrow & & \downarrow \\ \text{Witt} & \xrightarrow{\tau} & \mathfrak{sp}(H') \end{array}$$

from Figure 1. It is shown in [ADC, (5.11)] that the Lie algebras in (2.10) act transitively on the corresponding moduli spaces in (2.9), and the maps in (2.10) induce the differentials of the corresponding maps in (2.9).

2.7. The Hodge and theta line bundles. Here we review the degree-2 cohomology classes on the moduli spaces in consideration. Let $\Lambda \rightarrow \mathcal{A}_g$ be the *Hodge line bundle* defined by

$$\Lambda := \det(\pi_*(\omega_{\mathcal{X}_g/\mathcal{A}_g}))$$

where $\pi: \mathcal{X}_g \rightarrow \mathcal{A}_g$ is the natural projection and $\omega_{\mathcal{X}_g/\mathcal{A}_g}$ is the relative dualizing sheaf. Let $\Theta \rightarrow \mathcal{X}_g$ be the *relative theta line bundle* whose restriction to the fibers of π induces the principal polarization on each fiber. For simplicity, we omit the notation for the pull-back when referring to the pull-back of Λ and Θ via the nature projections $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ and $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$. Let λ and θ be the first Chern classes of Λ and Θ , respectively.

Proposition 2.4. *For $g \geq 2$, one has*

$$(2.11) \quad H^2(\widehat{\mathcal{A}}_g, \mathbb{Z}) = H^2(\mathcal{A}_g, \mathbb{Z}) = \mathbb{Z}\lambda,$$

$$(2.12) \quad H^2(\widehat{\mathcal{X}}_g, \mathbb{Z}) = H^2(\mathcal{X}_g, \mathbb{Z}) = \mathbb{Z}\lambda \oplus \mathbb{Z}\theta.$$

Moreover, for $g \geq 3$, the pull-back via the extended Torelli map induces an identification

$$(2.13) \quad H^2(\widehat{\mathcal{A}}_g, \mathbb{Z}) \xrightarrow{\cong} H^2(\widehat{\mathcal{M}}_g, \mathbb{Z}) = H^2(\mathcal{M}_g, \mathbb{Z})$$

and an inclusion

$$(2.14) \quad H^2(\widehat{\mathcal{X}}_g, \mathbb{Z}) \subset H^2(\widehat{\mathcal{P}}_{g-1}, \mathbb{Z}) = H^2(\mathcal{P}'_{g-1}, \mathbb{Z}).$$

The statement combines various results from the literature.

Proof. For $g \geq 2$, one has $H^2(\mathcal{A}_g, \mathbb{Z}) = \mathbb{Z}\lambda$, and for $g \geq 3$, the Torelli map induces an identification $H^2(\mathcal{A}_g, \mathbb{Z}) = H^2(\mathcal{M}_g, \mathbb{Z})$ (e.g., [VDG, 8.1] and [Hai, 17.3–17.4]). The last identity fails for $g = 2$, since $H^2(\mathcal{M}_2, \mathbb{Z})$ is generated by λ but cyclic of order 10.

Since \mathcal{A}_g and $\widehat{\mathcal{A}}_g$ have the same rational homotopy type [ADC, pg. 16–17], (2.11) follows. Similarly, \mathcal{X}_g and $\widehat{\mathcal{X}}_g$ have the same rational homotopy type [ADC, pg. 17], hence the first isomorphism in (2.12). The second identification in (2.12) follows from [FP].

Next, consider the composition of forgetful maps

$$(2.15) \quad \widehat{\mathcal{M}}_g \rightarrow \mathcal{M}'_g \rightarrow \mathcal{M}_g$$

where \mathcal{M}'_g is the moduli space of triples (C, P, v) , where C is a curve of genus g , P is a point of C , and v is a nonzero tangent vector to C at P . Since $\widehat{\mathcal{M}}_g$ and \mathcal{M}'_g have the same rational homotopy type and

$$H^2(\mathcal{M}'_g, \mathbb{Z}) = H^2(\mathcal{M}_g, \mathbb{Z}) = \mathbb{Z}\lambda$$

[ADCKP, 5.8], the composition of the maps in (2.15) induces an identification $H^2(\widehat{\mathcal{M}}_g, \mathbb{Z}) = \mathbb{Z}\lambda$. Using (2.11), we deduce (2.13). Note that the class of the cotangent line bundle at the marked point vanishes on \mathcal{M}'_g and $\widehat{\mathcal{M}}_g$.

Finally, $\widehat{\mathcal{P}}_{g-1}$ and \mathcal{P}'_{g-1} have the same rational homotopy type and $H^2(\mathcal{P}'_{g-1}, \mathbb{Z})$ is described in [ADCKP, 5.7]: it has dimension three, and λ and θ are linearly independent. The inclusion (2.14) follows, hence the statement. \square

3. GROUP (IND-)SCHEMES OF SYMPLECTIC AUTOMORPHISMS

Here we study how to exponentiate the symplectic algebra $\mathfrak{sp}(H')$ and related Lie algebras. We use group ind-schemes as in [FBZ, §A.1.4]. For a topological vector space E , let $\mathrm{GL}(E)$ be the group of continuous linear automorphisms of E . Also, for a topological group G , let G_\circ be its connected component of the identity.

3.1. The Lie algebra $\mathfrak{sp}^+(H')$. Let $\mathfrak{sp}^+(H')$ be the Lie subalgebra of $\mathfrak{sp}(H')$ which preserves the subspace H'_+ , that is:

$$\mathfrak{sp}^+(H') := \{X \in \mathfrak{sp}(H') \mid X(H'_+) \subseteq H'_+\}.$$

This is topologically generated by $S^2(H'_+)$ and the elements $b_i b_j \in \mathfrak{sp}(H')$ such that $i > 0 > j$.

The Lie algebra $\mathfrak{sp}^+(H')$ has the following geometric incarnation. Recall from (2.8) that for (Z, F, L) in $\widehat{\mathcal{A}}_g$, the tangent space to the fiber of the forgetful map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ at (Z, F, L) is $\widetilde{S}^2(H'_+) \times (H'_+ \otimes Z/F)$. One has a natural quotient of Lie algebras

$$(3.1) \quad \mathfrak{sp}^+(H') \twoheadrightarrow \widetilde{S}^2(H'_+) \times (H'_+ \otimes Z/F).$$

The transitive action of $\mathfrak{sp}(H')$ on $\widehat{\mathcal{A}}_g$ from §2.3 extends the transitive action of $\mathfrak{sp}^+(H')$ along the fibers of the forgetful map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. Hence, the symplectic uniformization of $\widehat{\mathcal{A}}_g$ induces the following double coset realization of the tangent space of \mathcal{A}_g . For $(Z, F, L) \in \widehat{\mathcal{A}}_g$, let $(X, B) \in \mathcal{A}_g$ be the image of (Z, F, L) via $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ as in (2.5). One has

$$(3.2) \quad T_{(X,B)}(\mathcal{A}_g) \cong \mathfrak{sp}_F(H') \setminus \mathfrak{sp}(H') / \mathfrak{sp}^+(H').$$

3.2. The group scheme $\mathcal{S}p^+(H')$. We will need to exponentiate the Lie algebra $\mathfrak{sp}^+(H')$. For this, consider the group scheme of continuous symplectic automorphisms of H' preserving the subspace H'_+ , and let $\mathcal{S}p^+(H')$ be its connected component of the identity, that is:

$$\mathcal{S}p^+(H') := \left\{ \rho \in \mathrm{GL}(H') \left| \begin{array}{l} \langle \rho(a), \rho(b) \rangle = \langle a, b \rangle \quad \text{for all } a, b \in H', \\ \rho(H'_+) \subseteq H'_+ \end{array} \right. \right\}_\circ.$$

The group scheme $\mathcal{S}p^+(H')$ represents the group functor which assigns to a \mathbb{C} -algebra R the identity component $\mathcal{S}p^+(H'(R))$ of the group of continuous symplectic automorphisms of $H'(R)$ preserving the subspace $H'_+(R) = tR[[t]]$.

One easily checks that

$$\mathrm{Lie}(\mathcal{S}p^+(H')) = \mathfrak{sp}^+(H').$$

Also, since $\mathcal{S}p^+(H')$ is by definition connected, elements in $\mathcal{S}p^+(H')$ are products of exponentials of elements in $\mathfrak{sp}^+(H')$.

The adjoint action of the group scheme $\mathcal{S}p^+(H')$ on $\mathfrak{sp}^+(H')$ induces a transitive action of $\mathcal{S}p^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$, and Theorem 2.2 as shown in [ADC] can be recasted as:

Theorem 3.1 (after [ADC]). *The moduli space $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}^+(H')$ compatible with the transitive action of $\mathcal{S}p^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$.*

Remark 3.2. As the group $\mathrm{GL}(H)$ is not connected [ADCKP, pg. 11], taking the identity component in the definition of $\mathcal{S}p^+(H')$ can not be omitted without modifying the outcome.

3.3. The group ind-scheme $\mathcal{S}p(H')$. More generally, consider the group functor which assigns to a \mathbb{C} -algebra R the connected group $\mathcal{S}p(H'(R))$ of continuous linear automorphisms of $H'(R)$ of the following type

$$\mathcal{S}p(H'(R)) := \left\{ \rho \in \mathrm{GL}(H'(R)) \left| \begin{array}{l} \langle \rho(a), \rho(b) \rangle = \langle a, b \rangle \quad \text{for all } a, b \in H', \\ \rho(H'_+(R)) \subseteq H'_+(R) \oplus H_-(R_n) \end{array} \right. \right\}_\circ.$$

where $R_n \subseteq R$ is the \mathbb{C} -subalgebra consisting of nilpotent elements. The presence of nilpotent elements entails that this functor can only be represented by an ind-scheme, as schemes are insufficient (this is as in [FBZ, §6.2.3]). Let $\mathcal{S}p(H')$ be the group ind-scheme representing this functor. Naturally, $\mathcal{S}p^+(H')$ is a subgroup of $\mathcal{S}p(H')$. One checks that

$$\mathrm{Lie}(\mathcal{S}p(H')) = \mathfrak{sp}(H').$$

3.4. Derivations and ring automorphisms. Here we remark how the Lie algebras and group (ind-)schemes of this section naturally extend analogous objects studied in [BD, FBZ] and related to the geometry of curves. Namely, let $\mathrm{Der}(\mathcal{O})$ be the functor assigning to a \mathbb{C} -algebra R the Lie algebra

$$\mathrm{Der}(\mathcal{O}(R)) = R[[t]]\partial_t.$$

This is the Lie subalgebra of the Lie algebra $\text{Witt}(R)$ from (1.6) topologically generated over R by L_p for $p \geq -1$. The inclusion $\tau: \text{Witt} \hookrightarrow \mathfrak{sp}(H')$ from (1.10) maps $\text{Der}(\mathcal{O})$ to $\mathfrak{sp}^+(H')$, so that one has a commutative diagram of Lie algebras

$$(3.3) \quad \begin{array}{ccc} \text{Witt} & \xleftarrow{\tau} & \mathfrak{sp}(H') \\ \uparrow & & \uparrow \\ \text{Der}(\mathcal{O}) & \hookrightarrow & \mathfrak{sp}^+(H'). \end{array}$$

From [ADCKP], the moduli space $\widehat{\mathcal{M}}_g$ carries a transitive action of Witt . This action is compatible with a simply transitive action of $\text{Der}(\mathcal{O})$ on the fibers of $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_g$: the elements L_p with $p \geq 0$ topologically generate the tangent directions to the changes of the formal coordinate at the marked point, and the element L_{-1} spans the tangent space to the curve at the marked point. Hence, for (C, P, t) in $\widehat{\mathcal{M}}_g$, one has

$$(3.4) \quad T_C(\mathcal{M}_g) \cong \text{Vect}(C \setminus P) \setminus \text{Witt} / \text{Der}(\mathcal{O}).$$

From [ADC], the moduli space $\widehat{\mathcal{A}}_g$ carries a transitive action of $\mathfrak{sp}(H')$ and by factoring Witt and $\mathfrak{sp}(H')$ by appropriate Lie subalgebras, the inclusion τ induces the differential of the extended Torelli map $\widehat{\mathcal{A}}_g \rightarrow \widehat{\mathcal{M}}_g$.

We emphasize how the transitive action of $\mathfrak{sp}^+(H')$ on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ is naturally the extension of the simply transitive action of $\text{Der}(\mathcal{O})$ on the fibers of $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_g$, and (3.2) extends (3.4).

One has a similar picture for the related group (ind-)schemes. Namely, consider first the group scheme $\text{Aut}(\mathcal{O})$ representing the functor which assigns to a \mathbb{C} -algebra R the group of continuous ring automorphisms of $R[[t]]$ preserving the ideal $tR[[t]]$. Explicitly, this is:

$$\text{Aut}(\mathcal{O}(R)) := \{t \mapsto a_1 t + a_2 t^2 + \dots \mid a_i \in R \text{ and } a_1 \text{ is a unit}\}.$$

Since the symplectic form (1.2) is invariant by changes of the parameter t and $tR[[t]] = H'_+(R)$ as vector spaces, it follows that $\text{Aut}(\mathcal{O})$ is a subgroup of $\mathfrak{Sp}^+(H')$. However, the Lie algebra of $\text{Aut}(\mathcal{O})$ is smaller than $\text{Der}(\mathcal{O})$: As remarked in [FBZ, §6.2.3], one has $\text{Lie}(\text{Aut}(\mathcal{O})) = t \text{Der}(\mathcal{O})$, and in order to exponentiate $\text{Der}(\mathcal{O})$, one needs to consider a group ind-scheme. For this, let $\underline{\text{Aut}}(\mathcal{O})$ be the group ind-scheme representing the functor which assigns to a \mathbb{C} -algebra R the group of continuous ring automorphisms of $R[[t]]$:

$$\underline{\text{Aut}}(\mathcal{O}(R)) := \left\{ t \mapsto a_0 + a_1 t + a_2 t^2 + \dots \mid \begin{array}{l} a_i \in R, a_1 \text{ is a unit,} \\ \text{and } a_0 \text{ is nilpotent} \end{array} \right\}.$$

Similarly, let $\text{Aut}(\mathcal{K})$ be the group ind-scheme representing the functor which assigns to a \mathbb{C} -algebra R the group of continuous ring automorphisms

of $R((t))$:

$$\mathrm{Aut}(\mathcal{K}(R)) := \left\{ t \mapsto \sum_{i \geq i_0} a_i t^i \mid \begin{array}{l} a_i \in R, a_1 \text{ is a unit,} \\ a_i \text{ nilpotent for } i \leq 0 \end{array} \right\}.$$

One has

$$\mathrm{Lie}(\underline{\mathrm{Aut}}(\mathcal{O})) = \mathrm{Der}(\mathcal{O}) \quad \text{and} \quad \mathrm{Lie}(\mathrm{Aut}(\mathcal{K})) = \mathrm{Witt}.$$

We refer to [BD, FBZ] for more on $\mathrm{Der}(\mathcal{O})$, $\underline{\mathrm{Aut}}(\mathcal{O})$ and $\mathrm{Aut}(\mathcal{K})$.

Elements in $\underline{\mathrm{Aut}}(\mathcal{O})$ and $\mathrm{Aut}(\mathcal{K})$ naturally induce elements in $\mathrm{GL}(H)$, and after composing with the projection $H \rightarrow H'$, they induce symplectic elements in $\mathrm{GL}(H')$. Hence, one has a commutative diagram of group (ind)-schemes

$$\begin{array}{ccc} \mathrm{Aut}(\mathcal{K}) & \hookrightarrow & \mathcal{S}p(H') \\ \uparrow & & \uparrow \\ \underline{\mathrm{Aut}}(\mathcal{O}) & \hookrightarrow & \mathcal{S}p^+(H') \end{array}$$

such that the differentials of these maps restricted to the tangent spaces at the identities are given by (3.3).

3.5. The group ind-scheme $\mathcal{M}p(H')$. We will also consider the group ind-scheme $\mathcal{M}p(H')$ given by the central extension

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathcal{M}p(H') \rightarrow \mathcal{S}p(H') \rightarrow 1$$

corresponding to the metaplectic algebra $\mathfrak{mp}(H')$ from (1.5).

3.6. The group scheme $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$. Finally, we describe the analogous Lie algebras and group (ind)-schemes for the moduli space $\widehat{\mathcal{X}}_g$. The transitive action of $\mathfrak{sp}(H') \times H'$ on $\widehat{\mathcal{X}}_g$ from §2.5 extends a transitive action of $\mathfrak{sp}^+(H') \times H'_+$ on the fibers of $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$. Let \mathcal{O}_1^\times be the group scheme of invertible Taylor series with constant term 1. This is a subgroup of the group scheme \mathcal{O}^\times of invertible Taylor series from [FBZ, §20.3.4]. One has $\mathrm{Lie}(\mathcal{O}^\times) = H_+$ and $\mathrm{Lie}(\mathcal{O}_1^\times) = H'_+$.

Theorem 3.3 (after [ADC]). *The moduli space $\widehat{\mathcal{X}}_g$ carries a transitive action of $\mathfrak{sp}(H') \times H'$ compatible with the transitive action of $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$ on the fibers of $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$.*

4. ON THE COHOMOLOGY OF THE LIE ALGEBRAS

Here we study the cohomology of the Lie algebras $\mathfrak{sp}(H')$ and $\mathfrak{sp}(H') \times H'$. We will then show in Theorem 5.1 that their H^2 spaces are canonically isomorphic to $H^2(\mathcal{A}_g, \mathbb{C})$ and $H^2(\mathcal{X}_g, \mathbb{C})$, respectively.

For a topological Lie algebra \mathfrak{g} , let $H^*(\mathfrak{g}, \mathbb{C})$ be its continuous Lie algebra cohomology with complex coefficients.

Recall the two-cocycle ψ of $\mathfrak{gl}(H)$ from (1.1), and consider the two-cocycles on $\mathfrak{sp}(H') \ltimes H'$

$$\begin{aligned}\alpha(X + f, Y + g) &:= \psi(X, Y), \\ \beta(X + f, Y + g) &:= \psi(f, g) = -\text{Res}_{t=0} f dg, \\ \gamma(X + f, Y + g) &:= \psi(X, g) - \psi(Y, f)\end{aligned}$$

for $X, Y \in \mathfrak{sp}(H')$ and $f, g \in H'$, where f, g act on H by multiplication. Hence, $\psi = \alpha + \beta + \gamma$ on $\mathfrak{sp}(H') \ltimes H'$. Restricting to $\text{Witt} \ltimes H'$, one has

$$\begin{aligned}\alpha(f\partial_t + g, h\partial_t + k) &= \frac{1}{6} \text{Res}_{t=0} f dh'', \\ \gamma(f\partial_t + g, h\partial_t + k) &= -\frac{1}{2} \text{Res}_{t=0} (f dk' - h dg').\end{aligned}$$

Similarly, consider the two-cocycle $\alpha(X, Y) := \psi(X, Y)$ on $\mathfrak{sp}(H')$. It is shown in [ADCKP, 2.1] that $H^1(\text{Witt}, \mathbb{C}) = H^1(\text{Witt} \ltimes H', \mathbb{C}) = 0$, and

$$H^2(\text{Witt}, \mathbb{C}) = \mathbb{C}\bar{\alpha} \quad \text{and} \quad H^2(\text{Witt} \ltimes H', \mathbb{C}) = \mathbb{C}\bar{\alpha} \oplus \mathbb{C}\bar{\beta} \oplus \mathbb{C}\bar{\gamma}$$

where $\bar{\alpha}$ is the class of α , and similarly for $\bar{\beta}$ and $\bar{\gamma}$. Recall $\tau: \text{Witt} \hookrightarrow \mathfrak{sp}(H')$ from (1.10) and $\sigma: \text{Witt} \ltimes H' \hookrightarrow \mathfrak{sp}(H') \ltimes H'$ from (1.12).

Proposition 4.1. *One has*

$$(4.1) \quad H^1(\mathfrak{sp}(H'), \mathbb{C}) = 0,$$

$$(4.2) \quad H^1(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) = 0,$$

$$(4.3) \quad H^2(\mathfrak{sp}(H'), \mathbb{C}) = \mathbb{C}\bar{\alpha},$$

$$(4.4) \quad H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) = \mathbb{C}\bar{\alpha} \oplus \mathbb{C}\bar{\beta}.$$

Moreover, the pull-back via τ induces the identification

$$\tau^*: H^2(\mathfrak{sp}(H'), \mathbb{C}) \xrightarrow{\cong} H^2(\text{Witt}, \mathbb{C}),$$

and the pull-back via σ induces the injection

$$\sigma^*: H^2(\mathfrak{sp}(H') \ltimes H', \mathbb{C}) \hookrightarrow H^2(\text{Witt} \ltimes H', \mathbb{C}), \quad \alpha \mapsto \alpha, \quad \beta \mapsto \frac{1}{2}\alpha + \psi.$$

Remark 4.2. The two-cocycle defining the central extension $\mathfrak{mp}(H')$ of $\mathfrak{sp}(H')$ in (1.6) is $-\frac{1}{2}\alpha$. Also, the two-cocycle defining the central extension $\widetilde{\mathcal{U}}_2(H)$ of $\mathfrak{sp}(H') \ltimes H'$ in (1.8) is $-\frac{1}{2}\alpha + \beta$.

Proof. Let H'_f be the functor which assigns to a \mathbb{C} -algebra R the R -subalgebra $H'_f(R) := R[t, t^{-1}] \subset H'(R)$. One checks that

$$\begin{aligned}S^2(H'_f) &= [S^2(H'_f), S^2(H'_f)] \quad \text{and} \\ S^2(H'_f) \ltimes H'_f &= [S^2(H'_f) \ltimes H'_f, S^2(H'_f) \ltimes H'_f].\end{aligned}$$

Since H'_f is dense in H' and $S^2(H')$ is dense in $\mathfrak{sp}(H')$, we deduce

$$\begin{aligned}\mathfrak{sp}(H') &= [\mathfrak{sp}(H'), \mathfrak{sp}(H')] \quad \text{and} \\ \mathfrak{sp}(H') \times H' &= [\mathfrak{sp}(H') \times H', \mathfrak{sp}(H') \times H']\end{aligned}$$

hence (4.1) and (4.2) follow.

Next we prove (4.4). Assume δ is an arbitrary two-cocycle on $\mathfrak{sp}(H') \times H'$. Recall the relations

$$(4.5) \quad [L_p, L_q] = (p - q) L_{p+q}, \quad [L_p, b_q] = -q b_{p+q}, \quad [b_p, b_q] = 0$$

on $\mathfrak{sp}(H') \times H'$, where $L_p = -t^{p+1} \partial_t$ and $b_q = t^q$ for $p, q \in \mathbb{Z}$. Considering terms of type

$$\delta(L_p, [L_q, L_r]), \quad \delta(L_p, [L_q, b_r]), \quad \delta(L_p, [b_q, b_r])$$

and imposing the two-cocycle relation

$$\delta(x, [y, z]) + \delta(y, [z, x]) + \delta(z, [x, y]) = 0$$

for all $x, y, z \in \mathfrak{sp}(H') \times H'$ and the relations (4.5), a direct computation as in the proof of [ADCKP, 2.1] shows that

$$\begin{aligned}\delta(X + f, Y + g) &\equiv A \alpha(X + f, Y + g) + B \beta(X + f, Y + g) \\ &\quad + C \gamma(X + f, Y + g)\end{aligned}$$

modulo coboundaries, for some $A, B, C \in \mathbb{C}$. Next, consider the two-cocycle relation for

$$x = b_1 b_1, \quad y = b_1 b_{-2}, \quad \text{and} \quad z = b_{-1}.$$

Observe that

$$[b_1 b_{-2}, b_{-1}] = b_{-2}, \quad [b_{-1}, b_1 b_1] = -2b_1, \quad [b_1 b_1, b_1 b_{-2}] = 0.$$

Since

$$\gamma(b_1 b_1, b_{-2}) = 2 \quad \text{and} \quad \gamma(b_1 b_{-2}, b_1) = 0,$$

it follows that the only non-trivial contribution to the two-cocycle relation is $C \gamma(b_1 b_1, [b_1 b_{-2}, b_{-1}]) = 0$, hence $C = 0$. A direct analysis shows that the two-cocycle relations do not impose any relations on A and B , hence (4.4).

The identification in (4.3) follows similarly from the argument for (4.4).

For the final part of the statement, recall the map $\widehat{\sigma}: \mathfrak{D} \hookrightarrow \widetilde{\mathcal{W}}_2(H)$ from (1.13). Since the two-cocycle on $\text{Witt} \times H'$ defining \mathfrak{D} is ψ , and the two-cocycle on $\mathfrak{sp}(H') \times H'$ defining $\widetilde{\mathcal{W}}_2(H)$ is $-\frac{1}{2}\alpha + \beta$, we deduce $\sigma^*(-\frac{1}{2}\alpha + \beta) = \psi$. A similar argument involving the map $\widehat{\tau}$ from (1.11) shows that $\sigma^*(\alpha) = \tau^*(\alpha) = \alpha$, hence the statement. \square

For a topological Lie algebra \mathfrak{g} , the space $H^2(\mathfrak{g}, \mathbb{C})$ classifies Lie algebra continuous central extensions of \mathfrak{g} up to isomorphism. Thus as a consequence of the previous statement, we have:

Proposition 4.3. *Let Z be a Lagrangian subspace of H' with $Z \cap H'_+ = 0$. Every Lie algebra continuous central extension*

$$0 \rightarrow \mathfrak{gl}_1 \rightarrow \widehat{\mathfrak{sp}(H')} \rightarrow \mathfrak{sp}(H') \rightarrow 0$$

—e.g., $\mathfrak{mp}(H')$ —splits over $\mathfrak{sp}_F(H')$ for all $F \subset Z$. Similarly, every Lie algebra continuous central extension

$$0 \rightarrow \mathfrak{gl}_1 \rightarrow \widehat{\mathfrak{sp}(H') \times H'} \rightarrow \mathfrak{sp}(H') \times H' \rightarrow 0$$

—e.g., $\widetilde{\mathcal{W}}_2(H)$ —splits over $\mathfrak{sp}_F(H') \times F$ for all $F \subset Z$.

Proof. One needs to show that the restriction of $H^2(\mathfrak{sp}(H'), \mathbb{C})$ to $\mathfrak{sp}_F(H')$ (respectively, the restriction of $H^2(\mathfrak{sp}(H') \times H', \mathbb{C})$ to $\mathfrak{sp}_F(H') \times F$) vanishes for all $F \subset Z$. From Proposition 4.1, it is enough to consider the class of the two-cocycle α (resp., the classes of the two-cocycles α and β). Also, after rescaling to $-\frac{1}{2}\alpha$ as in Remark 4.2, we can reduce to the case $\widehat{\mathfrak{sp}(H')} = \mathfrak{mp}(H')$.

For $Z = H_-$ and $F \subset Z$, one has $F \subset H_- \subset F^\perp$, thus for $X \in \mathfrak{sp}_F(H')$, one has

$$\mathrm{Im}(\pi_+ X \pi_-) = \pi_+ X(H_-) \subset \pi_+ X(F^\perp) \subseteq \pi_+ F = 0.$$

It follows that for $F \subset H_-$, one has $\alpha(X, Y) = 0$ for all X, Y in $\mathfrak{sp}_F(H')$. Next, consider an arbitrary Lagrangian subspace Z of H' with $Z \cap H'_+ = 0$ and a subspace $F \subset Z$. From §3.2, the group scheme $\mathcal{S}p^+(H')$ acts transitively on the space of such pairs (Z, F) , hence there exists $\rho \in \mathcal{S}p^+(H')$ such that $(Z, F) = \rho(H_-, \overline{F})$ for some $\overline{F} \subset H_-$. Let $\mathfrak{mp}_F(H')$ be the restriction of $\mathfrak{mp}(H')$ over $\mathfrak{sp}_F(H')$ and $\mathfrak{mp}_{\overline{F}}(H')$ the restriction over $\mathfrak{sp}_{\overline{F}}(H')$. The adjoint action of the group ind-scheme $\mathcal{M}p(H')$ on $\mathfrak{mp}(H')$ factors through $\mathcal{S}p(H')$, hence

$$\mathrm{Ad} \rho(\mathfrak{sp}_{\overline{F}}(H')) = \mathfrak{sp}_F(H') \quad \text{and} \quad \mathrm{Ad} \rho(\mathfrak{mp}_{\overline{F}}(H')) = \mathfrak{mp}_F(H').$$

Since we have shown that $\mathfrak{mp}_{\overline{F}}(H')$ is a trivial extension, it follows that $\mathfrak{mp}_F(H')$ is a trivial extension, hence $\alpha(X, Y) = 0$ for all X, Y in $\mathfrak{sp}_F(H')$.

The case of α on $\mathfrak{sp}(H') \times H'$ follows from α on $\mathfrak{sp}(H')$, and the case of β follows immediately from F being isotropic with respect to (1.2). \square

5. THE ISOMORPHISM OF SECOND COHOMOLOGY SPACES

Consider a variety X over \mathbb{C} carrying a transitive action of a Lie algebra \mathfrak{g} (as defined in §2.3). For $x \in X$, let $\mathfrak{g}_x := \mathrm{Ker}(\mathfrak{g} \rightarrow T_x(X))$. It is shown in [ADCKP, 4.1] that if $\mathfrak{g}_x = \overline{[\mathfrak{g}_x, \mathfrak{g}_x]}$ for all $x \in X$, then every Lie algebra continuous central extension

$$0 \rightarrow \mathbb{C} \rightarrow \widehat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$$

which splits over \mathfrak{g}_x for all $x \in X$ yields a continuous extension

$$0 \rightarrow X \times \mathbb{C} \rightarrow E \rightarrow T(X) \rightarrow 0.$$

This defines a canonical map

$$(5.1) \quad H^2(\mathfrak{g}, \mathbb{C})_0 \rightarrow \text{Ext}^1(\mathcal{T}_X, \mathcal{O}_X) = H^1(\Omega_X^1)$$

where $H^2(\mathfrak{g}, \mathbb{C})_0 \subset H^2(\mathfrak{g}, \mathbb{C})$ is the subspace obtained by intersecting the kernels of the restriction maps $H^2(\mathfrak{g}, \mathbb{C}) \rightarrow H^2(\mathfrak{g}_x, \mathbb{C})$ for all $x \in X$.

Combining Theorems 2.2 and 2.3 with Proposition 4.3, the above result from [ADCKP] applies to give canonical maps

$$\begin{aligned} \nu: H^2(\mathfrak{sp}(H'), \mathbb{C}) &\rightarrow H^2(\widehat{\mathcal{A}}_g, \mathbb{C}), \\ \mu: H^2(\mathfrak{sp}(H') \times H', \mathbb{C}) &\rightarrow H^2(\widehat{\mathcal{X}}_g, \mathbb{C}). \end{aligned}$$

We determine these maps explicitly in terms of the basis elements of the H^2 spaces given in Propositions 4.1 and 2.4:

Theorem 5.1. *For $g \geq 3$, one has*

$$(5.2) \quad \lambda = -\nu[\alpha],$$

$$(5.3) \quad \lambda = -\mu[\alpha], \quad \theta = \frac{1}{2} \mu[\alpha] - \mu[\beta].$$

In particular, the maps μ and ν give canonical isomorphisms

$$\begin{aligned} H^2(\mathfrak{sp}(H'), \mathbb{C}) &\cong H^2(\widehat{\mathcal{A}}_g, \mathbb{C}), \\ H^2(\mathfrak{sp}(H') \times H', \mathbb{C}) &\cong H^2(\widehat{\mathcal{X}}_g, \mathbb{C}). \end{aligned}$$

Proof. One has a commutative diagram

$$(5.4) \quad \begin{array}{ccc} H^2(\mathfrak{sp}(H'), \mathbb{C}) & \xrightarrow{\nu} & H^2(\widehat{\mathcal{A}}_g, \mathbb{C}) \\ \tau^* \downarrow \parallel & & \downarrow \parallel \\ H^2(\text{Witt}, \mathbb{C}) & \xrightarrow{\bar{\nu}} & H^2(\widehat{\mathcal{M}}_g, \mathbb{C}) \end{array}$$

where the map τ^* is the identification studied in Proposition 4.1, the map $\bar{\nu}$ follows from (5.1) and is an isomorphism by [ADCKP], and the right vertical map is the identification induced by the pull-back via the extended Torelli map as in Proposition 2.4. From [ADCKP, 4.10.iv], one has $\lambda = -\bar{\nu}[\alpha]$. Hence, (5.2) follows.

Furthermore, one has a commutative diagram

$$(5.5) \quad \begin{array}{ccc} H^2(\mathfrak{sp}(H') \times H', \mathbb{C}) & \xrightarrow{\mu} & H^2(\widehat{\mathcal{X}}_g, \mathbb{C}) \\ \sigma^* \downarrow & & \downarrow \\ H^2(\text{Witt} \times H', \mathbb{C}) & \xrightarrow{\bar{\mu}} & H^2(\widehat{\mathcal{P}}_{g-1}, \mathbb{C}) \end{array}$$

where the map σ^* is the injection studied in Proposition 4.1, the map $\bar{\mu}$ follows from (5.1) and is an isomorphism by [ADCKP], and the right vertical

map is the inclusion induced by the pull-back via the extended Torelli map as in Proposition 2.4. From [ADCKP, 4.10], one has $\lambda = -\bar{\mu}[\alpha]$ and $\theta = -\bar{\mu}[\psi]$. Applying the description of σ^* from Proposition 4.1, (5.3) follows. \square

Consequently, we obtain:

Proof of Theorem 5. The statement follows from Theorem 5.1 and Proposition 2.4. \square

Proof of Theorem 6. The statement follows from (5.2), (5.3), and Remark 4.2. \square

The spaces and line bundles appearing in Theorem 6 are summarized by the commutative diagram:

$$(5.6) \quad \begin{array}{ccccc} & & \Lambda & & \Theta \\ & \nearrow & \downarrow & \nearrow & \downarrow \\ \Lambda & & \widehat{\mathcal{A}}_g & & \Theta \\ \downarrow & \nearrow & \longleftarrow & \longleftarrow & \downarrow \\ \widehat{\mathcal{M}}_g & & \widehat{\mathcal{P}}_{g-1} & & \widehat{\mathcal{X}}_g \\ & \searrow & \longleftarrow & \searrow & \\ & & \widehat{\mathcal{P}}_{g-1} & & \end{array}$$

where the left and right squares are Cartesian. This diagram provides a geometric counterpoint to Figure 1. The conclusions of Theorem 6 are summarized by the following diagrams. Let $(Z, F, L) \rightarrow S$ be a family of extended abelian varieties over a smooth base S . Recall that Proposition 4.3 gives the splitting $\mathfrak{sp}_F(H') \hookrightarrow \mathfrak{mp}(H')$. Let \mathcal{F}_Λ be the Atiyah algebra of the line bundle Λ on $\widehat{\mathcal{A}}_g$, i.e., the sheaf of first-order differential operators acting on Λ . Theorem 6 implies the following commutative diagram of sheaves of Lie algebras with exact rows and columns:

$$(5.7) \quad \begin{array}{ccccc} & & \mathcal{O}_S & \xrightarrow{\frac{1}{2}} & \mathcal{O}_S \\ & & \downarrow & & \downarrow \\ \mathfrak{sp}_F(H'(\mathcal{O}_S)) & \hookrightarrow & \mathfrak{mp}(H'(\mathcal{O}_S)) & \twoheadrightarrow & \mathcal{F}_{\Lambda|S} \\ \downarrow \parallel & & \downarrow & & \downarrow \\ \mathfrak{sp}_F(H'(\mathcal{O}_S)) & \hookrightarrow & \mathfrak{sp}(H'(\mathcal{O}_S)) & \twoheadrightarrow & \mathcal{I}_S. \end{array}$$

Similarly, consider a family $(Z, F, L, \bar{h}, q) \rightarrow S$ over a smooth base S as in §2.4. Proposition 4.3 gives the splitting $\mathfrak{sp}_F(H') \times F \hookrightarrow \widehat{\mathcal{U}}_2(H)$. Let \mathcal{F}_Θ be the Atiyah algebra of the line bundle Θ on $\widehat{\mathcal{X}}_g$. Theorem 6 implies the

following commutative diagram of sheaves of Lie algebras with exact rows and columns:

$$(5.8) \quad \begin{array}{ccccc} & & \mathcal{O}_S & \xrightarrow{-1} & \mathcal{O}_S \\ & & \downarrow & & \downarrow \\ \mathrm{sp}_F(H'(\mathcal{O}_S)) \times F(\mathcal{O}_S) & \longleftarrow & \widetilde{\mathcal{U}}_2(H(\mathcal{O}_S)) & \longrightarrow & \mathcal{F}_{\Theta|S} \\ & & \downarrow & & \downarrow \\ \mathrm{sp}_F(H'(\mathcal{O}_S)) \times F(\mathcal{O}_S) & \longleftarrow & \mathrm{sp}(H'(\mathcal{O}_S)) \times H'(\mathcal{O}_S) & \longrightarrow & \mathcal{I}_S. \end{array}$$

6. INTERMEZZO ON METAPLECTIC REPRESENTATIONS

Here we define some properties of representations of the metaplectic algebra which will be used in the following sections.

Definition 6.1. (i) An *admissible* $\mathfrak{mp}(H'(\mathbb{C}))$ representation V is a representation of $\mathfrak{mp}(H'(\mathbb{C}))$ such that:

- (a) the action of $b_{-i} b_i$ on V is diagonalizable with integral eigenvalues for $i \geq 1$, and
- (b) the action of $b_i b_j$ on V is locally nilpotent for i, j not both negative and $i + j \neq 0$.

(ii) An *admissible* $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representation V is a representation of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ which induces an *admissible* $\mathfrak{mp}(H'(\mathbb{C}))$ representation on V and such that the action of $H'_+ \subset \widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ is locally nilpotent on V .

We will also use the following induced characterization. Consider that a representation V of the Virasoro algebra is said to be of central charge $c \in \mathbb{C}$ if the actions of the Virasoro operators on V satisfy

$$[L_p, L_q] = (p - q) L_{p+q} + \frac{c}{12} (p^3 - p) \delta_{p+q,0} \mathrm{id}_V \quad \text{for } p, q \in \mathbb{Z}.$$

Recall the inclusion (1.11).

Definition 6.2. A representation V of $\mathfrak{mp}(H'(\mathbb{C}))$ (respectively, $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$) is said to be of *central charge* $c \in \mathbb{C}$ if the action of $\mathfrak{mp}(H'(\mathbb{C}))$ (resp., $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$) induces an action of Vir on V of central charge c .

6.1. Examples. Examples of admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representations are given by certain vertex operator algebras for which the action of the Virasoro algebra extends to an action of the metaplectic algebra. We refer to [FHL, Kac] for treatments of vertex operator algebras. Briefly, a vertex operator algebra is a $\mathbb{Z}_{\geq 0}$ -graded complex vector space V together with a distinguished degree-0 element $\mathbf{1}$, a distinguished degree-2 element ω , and a linear map $Y(\cdot, t): V \rightarrow \mathrm{End}(V)[[t, t^{-1}]]$, satisfying suitable conditions. The Fourier coefficients of $Y(\omega, t)$ realize an action of the Virasoro algebra on V .

Via the inclusion (1.11), the Virasoro element L_p is realized in the metaplectic algebra as $\frac{1}{2} \sum_i : b_{-i} b_{i+p} :$. The element L_0 acts on a vertex operator algebra V as the grading operator, and the elements L_p with $p > 0$ have negative degree on V . Consequently, the action of $\frac{1}{2} \sum_i : b_{-i} b_i :$ on V is diagonalizable with integral eigenvalues and the action of $\frac{1}{2} \sum_i : b_{-i} b_{i+p} :$ on V with $p > 0$ is locally nilpotent. It follows that the conditions defining admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representations in Definition 6.1 provide a strengthening of these properties.

For instance, we have:

Lemma 6.3. *Heisenberg vertex algebras of arbitrary rank and even lattice vertex algebras are admissible $\widehat{\mathcal{U}}_2(H(\mathbb{C}))$ representations.*

Proof. Let V be either the rank-one Heisenberg vertex algebra or a rank-one even lattice vertex algebra. The degree of the operator b_i on V is $-i$, hence the degree of $b_i b_j$ on V is $-(i+j)$. In particular, H'_+ and $\widetilde{S}^2(H'_+)$ act on V by operators of negative degree, hence H'_+ and $\widetilde{S}^2(H'_+)$ act locally nilpotently on V .

Next, consider $b_i b_j$ with $i > 0$ and $j < 0$. If $i+j > 0$, then the degree of $b_i b_j$ on V is still negative, hence $b_i b_j$ acts locally nilpotently on V . If $i+j < 0$, then the degree of $b_i b_j$ on V is positive; however, b_i and b_j commute, and the degree of b_i is negative, hence $b_i b_j$ still acts locally nilpotently on V .

Finally, it is easy to see that the action of $b_{-i} b_i$ on V for $i \geq 1$ is diagonalizable with integral eigenvalues.

The case of arbitrary rank is similar. □

7. COINVARIANTS ON FAMILIES OF EXTENDED ABELIAN VARIETIES

We define here spaces of coinvariants at extended abelian varieties and show how these yield twisted \mathcal{D} -modules on $\widehat{\mathcal{A}}_g$. Using results from §5, we identify a multiple of the Atiyah algebra of the line bundle Λ on $\widehat{\mathcal{A}}_g$ which acts on the sheaves of coinvariants and thus determines their twisted \mathcal{D} -module structure. Similarly, we define and study twisted \mathcal{D} -modules of coinvariants on $\widehat{\mathcal{X}}_g$.

7.1. Spaces of coinvariants at extended PPAVs. Let (Z, F, L) be an extended abelian variety over a \mathbb{C} -algebra R . Consider a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$. The action of $\mathfrak{mp}(H'(\mathbb{C}))$ on V extends R -linearly to an action of $\mathfrak{mp}(H'(R))$ on $V \otimes_{\mathbb{C}} R$. Composing with the splitting $\mathfrak{sp}_F(H') \hookrightarrow \mathfrak{mp}(H')$ from Proposition 4.3, one has an action of $\mathfrak{sp}_F(H'(R))$ on $V \otimes_{\mathbb{C}} R$. We define the *space of coinvariants* of V at (Z, F, L) as

$$\widehat{V}(V)_{(Z,F,L)} := V \otimes_{\mathbb{C}} R / \mathfrak{sp}_F(H'(R))(V \otimes_{\mathbb{C}} R).$$

7.2. Sheaves of coinvariants on $\widehat{\mathcal{A}}_g$. The spaces of coinvariants induce sheaves of coinvariants as follows. Let $(Z, F, L) \rightarrow S$ be a family of extended abelian varieties over a smooth base S . Consider a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$. The action of $\mathfrak{mp}(H'(\mathbb{C}))$ on V extends \mathcal{O}_S -linearly to an action of the sheaf of Lie algebras $\mathfrak{mp}(H'(\mathcal{O}_S))$ on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_S$. This action restricts to an action of the sheaf of Lie algebras $\mathfrak{sp}_F(H'(\mathcal{O}_S))$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ via the splitting $\mathfrak{sp}_F(H'(\mathcal{O}_S)) \hookrightarrow \mathfrak{mp}(H'(\mathcal{O}_S))$ as in (5.7). We define the *sheaf of coinvariants* of V on $(Z, F, L) \rightarrow S$ as the quasi-coherent sheaf of \mathcal{O}_S -modules

$$\widehat{\mathbb{V}}(V)_{(Z,F,L) \rightarrow S} := V \otimes_{\mathbb{C}} \mathcal{O}_S / \mathfrak{sp}_F(H'(\mathcal{O}_S))(V \otimes_{\mathbb{C}} \mathcal{O}_S).$$

This gives rise to a quasi-coherent sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$.

Theorem 7.1. *For a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$ of central charge c , the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an action of the Atiyah algebra $\frac{c}{2} \mathcal{F}_\Lambda$. This action induces a twisted \mathcal{D} -module structure on $\widehat{\mathbb{V}}(V)$.*

Proof. The action of $\mathfrak{mp}(H'(\mathbb{C}))$ on V and the action of $\mathfrak{mp}(H')$ on $\widehat{\mathcal{A}}_g$ via the projection $\mathfrak{mp}(H') \rightarrow \mathfrak{sp}(H')$ induce an action of the sheaf of Lie algebras $\mathfrak{mp}(H'(\mathcal{O}_S))$ on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_S$. While it is not \mathcal{O}_S -linear, this action extends the \mathcal{O}_S -linear action of $\mathfrak{sp}_F(H'(\mathcal{O}_S))$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ since $\mathfrak{sp}_F(H')$ acts trivially on $\widehat{\mathcal{A}}_g$. Applying the central row of (5.7), the action of $\mathfrak{mp}(H'(\mathcal{O}_S))$ factors to an action of the Atiyah algebra $\alpha \mathcal{F}_\Lambda$ on $\widehat{\mathbb{V}}(V)$, for some $\alpha \in \mathbb{C}$. Since the central element $\mathbf{1}$ of $\mathfrak{mp}(H')$ acts on V as multiplication by the central charge c , the first row of (5.7) implies $\alpha = \frac{c}{2}$, hence the statement. \square

7.3. Sheaves of coinvariants on $\widehat{\mathcal{X}}_g$. Let $(Z, F, L, \bar{h}, q) \rightarrow S$ be a family as in §2.4 over a smooth base S . Consider a representation V of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$. The action of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ on V extends \mathcal{O}_S -linearly to an action of the sheaf of Lie algebras $\widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$ on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_S$. Composing with the splitting

$$\mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S) \hookrightarrow \widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$$

as in (5.8), one has an action of $\mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S)$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$. We define the *sheaf of coinvariants* of V on $(Z, F, L, \bar{h}, q) \rightarrow S$ as the quasi-coherent sheaf of \mathcal{O}_S -modules

$$\widehat{\mathbb{V}}(V)_{(Z,F,L,\bar{h},q) \rightarrow S} := V \otimes_{\mathbb{C}} \mathcal{O}_S / (\mathfrak{sp}_F(H'(\mathcal{O}_S)) \ltimes F(\mathcal{O}_S))(V \otimes_{\mathbb{C}} \mathcal{O}_S).$$

This gives rise to a quasi-coherent sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$.

Theorem 7.2. *For a representation V of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ of central charge c , the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ carries an action of the Atiyah algebra $-c \mathcal{F}_\Theta$. This action induces a twisted \mathcal{D} -module structure on $\widehat{\mathbb{V}}(V)$.*

Proof. The action of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ on V and the action of $\widetilde{\mathcal{U}}_2(H)$ on $\widehat{\mathcal{X}}_g$ via the projection $\widetilde{\mathcal{U}}_2(H) \rightarrow \mathfrak{sp}(H') \ltimes H'$ induce an action of the sheaf of Lie algebras $\widetilde{\mathcal{U}}_2(H(\mathcal{O}_S))$ on the sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_S$. Applying the central row of (5.8), this action factors to an action of the Atiyah algebra ${}_{\alpha} \mathcal{F}_{\Theta}$ on $\widehat{\mathbb{V}}(V)$, for some $\alpha \in \mathbb{C}$. Since the central element $\mathbf{1}$ of $\widetilde{\mathcal{U}}_2(H)$ acts on V as multiplication by c , the first row of (5.8) implies $\alpha = -c$, hence the statement. \square

8. COINVARIANTS ON FAMILIES OF ABELIAN VARIETIES

Here we construct twisted \mathcal{D} -modules of coinvariants on \mathcal{A}_g by descending the twisted \mathcal{D} -modules $\widehat{\mathbb{V}}(V)$ from §7 along the projection $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. We proceed similarly on \mathcal{X}_g .

We use the following splittings. It is immediate to see that the two-cocycle on $\mathfrak{sp}(H')$ defining $\mathfrak{mp}(H')$ from (1.6) vanishes on the Lie algebra $\mathfrak{sp}^+(H')$ from §3.1, hence one has a Lie algebra splitting

$$(8.1) \quad \mathfrak{sp}^+(H') \hookrightarrow \mathfrak{mp}(H').$$

Similarly, the two-cocycle on $\mathfrak{sp}(H') \ltimes H'$ defining $\widetilde{\mathcal{U}}_2(H)$ from (1.8) vanishes on $\mathfrak{sp}^+(H') \ltimes H'_+$, hence one has a Lie algebra splitting

$$(8.2) \quad \mathfrak{sp}^+(H') \ltimes H'_+ \hookrightarrow \widetilde{\mathcal{U}}_2(H).$$

8.1. The action of $\mathcal{S}p^+(H')$. Let V be a representation of $\mathfrak{mp}(H'(\mathbb{C}))$ of central charge c . From Theorem 7.1, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an action of the Atiyah algebra ${}_{\frac{c}{2}} \mathcal{F}_{\Lambda}$. This is induced by an action of $\mathfrak{mp}(H')$ on $\widehat{\mathbb{V}}(V)$ via the projection $\mathfrak{mp}(H') \rightarrow {}_{\frac{c}{2}} \mathcal{F}_{\Lambda}$. Composing with the inclusion (8.1), we deduce an action of $\mathfrak{sp}^+(H')$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$. Next, we show:

Proposition 8.1. *For an admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation V , the action of $\mathfrak{sp}^+(H')$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ can be exponentiated to an equivariant action of $\mathcal{S}p^+(H')$ on $\widehat{\mathbb{V}}(V)$.*

Proof. Recall from §3.2 that since $\mathcal{S}p^+(H')$ is connected, elements in $\mathcal{S}p^+(H')$ are products of exponentials of elements in $\mathfrak{sp}^+(H')$. Hence it is enough to show that the action on $\widehat{\mathbb{V}}(V)$ of exponentials of elements in $\mathfrak{sp}^+(H')$ is well-defined.

By definition, $\mathfrak{sp}^+(H')$ is topologically generated by elements of type $b_i b_j \in S^2(H')$ with i, j not both negative. First consider $b_i b_j$ with $i + j = 0$. Since V is an admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation, the action of $b_i b_j$ on V is diagonalizable with integral eigenvalues by Definition 6.1. This action can be exponentiated to an action of the multiplicative group scheme \mathbb{G}_m on V by letting $a \in \mathbb{G}_m$ act as multiplication by a^k on the eigenspace of $b_i b_j$ with eigenvalue k .

Next, consider $b_i b_j$ with i, j not both negative and $i + j \neq 0$. By Definition 6.1, the action of $b_i b_j$ on V is locally nilpotent and hence can be

exponentiated, since the formula for the action of the exponential of $b_i b_j$ is locally a finite sum.

Finally, we discuss the case of an arbitrary element of $\mathfrak{sp}^+(H')$. For (Z, F, L) in $\widehat{\mathcal{A}}_g$, Theorem 2.2 implies that $\mathfrak{sp}_F(H')$ is preserved by the action of $\mathfrak{sp}^+(H')$. After quotienting by $\mathfrak{sp}_F(H')$, the Lie algebra $\mathfrak{sp}^+(H')$ factors to $\widetilde{S}^2(H'_+) \times (H'_+ \otimes Z/F)$ as in (3.1), thus the action of $\mathfrak{sp}^+(H')$ on $V \otimes_{\mathbb{C}} \mathcal{O}_S$ factors to an action of $\widetilde{S}^2(H'_+) \times (H'_+ \otimes Z/F)$ on $\widehat{\mathbb{V}}(V)$. This implies that for an arbitrary element of $\mathfrak{sp}^+(H')$, only finitely many terms $b_i b_j$ with $i+j \leq 0$ act locally non-trivially on $\widehat{\mathbb{V}}(V)$. Using the arguments in the previous paragraphs, one checks that the formula for the action of exponentials of elements in $\mathfrak{sp}^+(H')$ with only finitely many terms $b_i b_j$ with $i+j \leq 0$ is locally a finite sum on $\widehat{\mathbb{V}}(V)$. The statement follows. \square

8.2. Sheaves of coinvariants on \mathcal{A}_g . Let V be an admissible $\mathfrak{mp}(H'(\mathbb{C}))$ representation. From Proposition 8.1, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an equivariant action of $\mathcal{S}p^+(H')$. From Theorem 3.1, $\mathcal{S}p^+(H')$ acts transitively on the fibers of the map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$. It follows that the quotient of $\widehat{\mathbb{V}}(V)$ by the action of $\mathcal{S}p^+(H')$ descends along the projection $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ to a sheaf on \mathcal{A}_g , which we call the *sheaf of coinvariants* $\mathbb{V}(V)$ on \mathcal{A}_g . One has a Cartesian diagram

$$(8.3) \quad \begin{array}{ccc} \widehat{\mathbb{V}}(V) & \longrightarrow & \mathbb{V}(V) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{A}}_g & \longrightarrow & \mathcal{A}_g. \end{array}$$

Applying Theorem 7.1, we are now ready for:

Proof of Theorems 1 and 2. From Theorem 7.1, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$ carries an action of the Atiyah algebra $\frac{c}{2} \mathcal{F}_\Lambda$. Since this action is compatible with the action of $\mathcal{S}p^+(H')$, it induces an action of the Atiyah algebra $\frac{c}{2} \mathcal{F}_\Lambda$ on the sheaf $\mathbb{V}(V)$ on \mathcal{A}_g , hence the statements. \square

Example 8.2. When $c = 0$, the action of the Atiyah algebra factors to an action of the tangent sheaf to \mathcal{A}_g on the sheaf of coinvariants $\mathbb{V}(V)$ on \mathcal{A}_g . Hence for $c = 0$, the sheaf $\mathbb{V}(V)$ is more simply a \mathcal{D} -module on \mathcal{A}_g .

8.3. The action of $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$. Next, we discuss how to construct similarly sheaves of coinvariants on \mathcal{X}_g . For this, we start describing the action of $\mathcal{S}p^+(H') \ltimes \mathcal{O}_1^\times$ on the sheaf of coinvariants on $\widehat{\mathcal{X}}_g$.

Let V be a representation of $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ of central charge c . From Theorem 7.2, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ carries an action of the Atiyah algebra $-c \mathcal{F}_\Theta$. This is induced by an action of $\widetilde{\mathcal{U}}_2(H)$ on $\widehat{\mathbb{V}}(V)$ via the projection $\widetilde{\mathcal{U}}_2(H) \rightarrow -c \mathcal{F}_\Lambda$. Composing with the inclusion (8.2), we deduce an action of $\mathfrak{sp}^+(H') \ltimes H'_+$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$.

Proposition 8.3. *For an admissible $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representation V , the action of $\mathfrak{sp}^+(H') \times H'_+$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ can be exponentiated to an action of $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$ on $\widehat{\mathbb{V}}(V)$.*

The statement follows similarly to Proposition 8.1, since elements in $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$ are products of exponentials of elements in $\mathfrak{sp}^+(H') \times H'_+$, and H'_+ acts locally nilpotently on V by Definition 6.1.

8.4. Sheaves of coinvariants on \mathcal{X}_g . For an admissible $\widetilde{\mathcal{U}}_2(H(\mathbb{C}))$ representation V , Proposition 8.3 gives an equivariant action of $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$ on the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{A}}_g$. Since from Theorem 3.3, $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$ acts transitively on the fibers of the map $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$, it follows that $\widehat{\mathbb{V}}(V)$ descends along the projection $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$ to a sheaf on \mathcal{X}_g , which we call the *sheaf of coinvariants* $\mathbb{V}(V)$ on \mathcal{X}_g . One has a Cartesian diagram as in (8.3), with $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ replaced by $\widehat{\mathcal{X}}_g \rightarrow \mathcal{X}_g$.

Proof of Theorems 3 and 4. From Theorem 7.2, the sheaf $\widehat{\mathbb{V}}(V)$ on $\widehat{\mathcal{X}}_g$ carries an action of the Atiyah algebra $-c\mathcal{F}_\Theta$. Since this action is compatible with the action of $\mathcal{S}p^+(H') \times \mathcal{O}_1^\times$, it induces an action of the Atiyah algebra $-c\mathcal{F}_\Theta$ on the sheaf $\mathbb{V}(V)$ on \mathcal{X}_g , hence the statements. \square

9. FINAL REMARKS

9.1. Geography of Harish-Chandra pairs. The constructions of the sheaves of coinvariants in §8 are variations on the formalism of localization of modules over Harish-Chandra pairs first introduced in [BB] and further developed in [BFM, BS] and [FBZ, §17.2]. A Harish-Chandra pair (\mathfrak{g}, K) consists of a Lie algebra \mathfrak{g} and a Lie group K verifying some compatibility conditions. The localization functor assigns to a module over (\mathfrak{g}, K) , i.e., a vector space with compatible actions of \mathfrak{g} and K , a (possibly twisted) \mathcal{D} -module on a variety identified by the Harish-Chandra pair. For instance, localizations of: modules over $(\text{Der}(\mathcal{O}), \text{Aut}(\mathcal{O}))$ (see notation from §3.4) yield \mathcal{D} -modules on a smooth curve; modules over $(\text{Witt}, \text{Aut}(\mathcal{O}))$ yield \mathcal{D} -modules on the moduli space $\mathcal{M}_{g,1}$ of pointed smooth curves of genus g ; and modules over $(\text{Vir}, \text{Aut}(\mathcal{O}))$ yield twisted \mathcal{D} -modules on $\mathcal{M}_{g,1}$. We refer to [FBZ] for a discussion of these and more geometries.

The construction of the twisted \mathcal{D} -modules of coinvariants on \mathcal{A}_g and \mathcal{X}_g in §8 can be interpreted as the result of the localization of modules over the Harish-Chandra pairs

$$(\mathfrak{mp}(H'), \mathcal{S}p^+(H')) \quad \text{and} \quad \left(\widetilde{\mathcal{U}}_2(H), \mathcal{S}p^+(H') \times \mathcal{O}_1^\times \right),$$

respectively. We emphasize a difference here with respect to the geometry of curves: While the group $\text{Aut}(\mathcal{O})$ acts simply transitively on the fibers of $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_{g,1}$, the group $\mathcal{S}p^+(H')$ acts transitively, but not simply, on the fibers of $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$, as discussed in §3.

9.2. Comparison with classical coinvariants on curves. For a vertex operator algebra V , the space of coinvariants at $(C, P, t) \in \widehat{\mathcal{M}}_g$ constructed in [FBZ] is the quotient of V by the action of a Lie algebra $\mathcal{L}_{C \setminus P}(V)$ depending on both V and the open subset $C \setminus P \subset C$. The kernel of the map $\text{Witt} \rightarrow T_{(C, P, t)}(\widehat{\mathcal{M}}_g)$ is a Lie subalgebra of $\mathcal{L}_{C \setminus P}(V)$, but in general is not equal to it. It is shown in [FBZ] how the construction globalizes over $\widehat{\mathcal{M}}_g$ and yields an $\text{Aut}(\mathcal{O})$ -equivariant twisted \mathcal{D} -module on $\widehat{\mathcal{M}}_g$. This thus descends along the principal $\text{Aut}(\mathcal{O})$ -bundle $\widehat{\mathcal{M}}_g \rightarrow \mathcal{M}_{g,1}$. Furthermore, since the resulting fibers at (C, P) and (C, Q) in $\mathcal{M}_{g,1}$ are canonically isomorphic for all $P, Q \in C$, the sheaf is constant along the fibers of $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ and thus descends to a twisted \mathcal{D} -module on \mathcal{M}_g .

For a representation V of $\mathfrak{mp}(H'(\mathbb{C}))$, the space of coinvariants at $(Z, F, L) \in \widehat{\mathcal{A}}_g$ from §7 is the quotient of V by the action of the Lie algebra $\mathfrak{sp}_F(H')$. Contrary to $\mathcal{L}_{C \setminus P}(V)$, the Lie algebra $\mathfrak{sp}_F(H')$ is independent of V and equals the kernel of the analogous map $\mathfrak{sp}(H') \rightarrow T_{(Z, F, L)}(\widehat{\mathcal{A}}_g)$, hence it is the smallest Lie algebra whose coinvariants yield twisted \mathcal{D} -modules on $\widehat{\mathcal{A}}_g$. This implies that the push-forward via the Torelli injection $\mathcal{M}_g \rightarrow \mathcal{A}_g$ of the sheaf of coinvariants for $\mathcal{L}_{C \setminus P}(V)$ on \mathcal{M}_g does not map to the sheaf of coinvariants for $\mathfrak{sp}_F(H')$ on \mathcal{A}_g for a general V . For this, it is natural to determine a Lie algebra containing as Lie subalgebras both $\mathfrak{sp}_F(H')$ and $\mathcal{L}_{C \setminus P}(V)$ at points in the Jacobian locus. We will present such an extension in a follow-up work.

The spaces of coinvariants for $\mathcal{L}_{C \setminus P}(V)$ are known to have finite dimension when V satisfies some finiteness and semisimplicity conditions [AN, DGT2], and thus give rise to vector bundles of coinvariants on \mathcal{M}_g whose Chern classes are determined by their twisted \mathcal{D} -module structure [DGT1]. It would be interesting to determine a similar result on spaces of coinvariants on \mathcal{A}_g . The mentioned extension of $\mathfrak{sp}_F(H')$ will provide a better context for investigating this property, which we intend to explore accordingly.

9.3. Failing descent for arbitrary modules. Spaces of coinvariants for $\mathcal{L}_{C \setminus P}(V)$ have been constructed also for the action of $\mathcal{L}_{C \setminus P}(V)$ on an arbitrary vertex operator algebra module [FBZ]. These give rise to twisted \mathcal{D} -modules on $\mathcal{M}_{g,1}$ [DGT2, §8.7], which in general do not descend on \mathcal{M}_g . In fact, the descent along $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ is only possible after tensoring the sheaf of coinvariants by an appropriate power of the relative cotangent line bundle to offset the variation of the spaces of coinvariants along the fibers of $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$ (as in [DGT2, §8.7]). Similarly, the twisted \mathcal{D} -modules of coinvariants from arbitrary vertex operator algebra modules constructed as in §7 in general do not descend on \mathcal{A}_g . As the relative H^2 -space for the map $\widehat{\mathcal{A}}_g \rightarrow \mathcal{A}_g$ is trivial (Proposition 2.4), tensoring by a line bundle will not help here. It would be interesting to find a finite-dimensional extension of the moduli space \mathcal{A}_g where the sheaves of coinvariants from arbitrary vertex operator algebra modules could descend from $\widehat{\mathcal{A}}_g$.

ACKNOWLEDGMENTS

The author acknowledges partial support from the Simons Foundation’s Travel Support for Mathematicians gift during the preparation of this manuscript.

REFERENCES

- [ADC] E. Arbarello and C. De Concini. Abelian varieties, infinite dimensional Lie algebras, and the heat equation. In *Proceedings of Symposia in Pure Mathematics*, volume 53, 1991. ↑1, ↑2, ↑4, ↑8, ↑9, ↑10, ↑11, ↑12, ↑14, ↑15, ↑16
- [ADCK] E. Arbarello, C. De Concini, and V. Kac. The infinite wedge representation and the reciprocity law for algebraic curves. In *Theta functions—Bowdoin*, volume 49, Part 1, pages 171–190. American Mathematical Society, 1987. ↑4
- [ADCKP] E. Arbarello, C. De Concini, V. G. Kac, and C. Procesi. Moduli spaces of curves and representation theory. *Comm. Math. Phys.*, 117(1):1–36, 1988. ↑3, ↑11, ↑13, ↑14, ↑15, ↑17, ↑18, ↑19, ↑20, ↑21
- [AN] T. Abe and K. Nagatomo. Finiteness of conformal blocks over compact Riemann surfaces. *Osaka J. Math.*, 40(2):375–391, 2003. ↑28
- [BB] A. Beilinson, A. and J. Bernstein. A proof of Jantzen conjectures. In *I. M. Gelfand Seminar*, volume 16 of *Adv. Soviet Math.*, pages 1–50. Amer. Math. Soc., Providence, RI, 1993. ↑27
- [BD] A. Beilinson and V. Drinfeld. Quantization of Hitchin’s integrable system and Hecke eigensheaves. *Preprint, available at <http://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf>*, 1991. ↑3, ↑14, ↑16
- [BFM] A. Beilinson, B. Feigin, and B. Mazur. Introduction to algebraic field theory on curves. *Unpublished*, 1991. ↑27
- [BS] A. A. Beilinson and V. V. Schechtman. Determinant bundles and Virasoro algebras. *Comm. Math. Phys.*, 118(4):651–701, 1988. ↑2, ↑27
- [BZN] D. Ben-Zvi and T. Nevins. \mathcal{W} -symmetry of the adèlic Grassmannian. *Communications in Mathematical Physics*, 293(1):185–204, 2010. ↑1
- [DGT1] C. Damiolini, A. Gibney, and N. Tarasca. Conformal blocks from vertex algebras and their connections on $\overline{\mathcal{M}}_{g,n}$. *Geometry & Topology*, 25(5):2235–2286, 2021. ↑28
- [DGT2] C. Damiolini, A. Gibney, and N. Tarasca. On factorization and vector bundles of conformal blocks from vertex algebras. *Annales Scientifiques de l’École Normale Supérieure, to appear, arXiv:1909.04683*, 2022. ↑28
- [FBZ] E. Frenkel and D. Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. Am Mathem Soc, Providence, RI, second edition, 2004. ↑1, ↑3, ↑6, ↑7, ↑13, ↑14, ↑15, ↑16, ↑27, ↑28
- [FHL] I. B. Frenkel, Y.-Z. Huang, and J. Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.*, 104(494):viii+64, 1993. ↑22
- [FP] R. Fringuelli and R. Pirisi. The Picard group of the universal abelian variety and the Franchetta conjecture for abelian varieties. *Michigan Mathematical Journal*, 68(3):651–671, 2019. ↑12
- [FS] E. Frenkel and M. Szczesny. Twisted modules over vertex algebras on algebraic curves. *Advances in Mathematics*, 187(1):195–227, 2004. ↑1
- [Hai] R. Hain. Moduli of Riemann surfaces, transcendental aspects. In *School on Algebraic Geometry (Trieste, 1999)*, pages 293–353. Abdus Salam Int. Cent. Theoret. Phys., Trieste. ↑12
- [Kac] V. Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. Am Mathem Soc, Providence, RI, 1997. ↑22

- [TUY] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 459–566. Academic Press, Boston, MA, 1989. ↑1
- [VDG] G. Van Der Geer. The cohomology of the moduli space of abelian varieties. In *Handbook of Moduli I*, volume 24 of *Advanced Lectures in Mathematics*, pages 415–458. International Press of Boston, Inc., 2013. ↑12

NICOLA TARASCA
DEPARTMENT OF MATHEMATICS & APPLIED MATHEMATICS
VIRGINIA COMMONWEALTH UNIVERSITY, RICHMOND, VA 23284
Email address: `tarascan@vcu.edu`