Section 7.2 Linear Homogeneous Recurrence Relations (Continued)

Recall our main theorem for solving linear homogeneous recur. rel.

Theorem 7.2.1

Consider a homogeneous linear recurrence relation

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0.$$

Suppose the characteristic equation

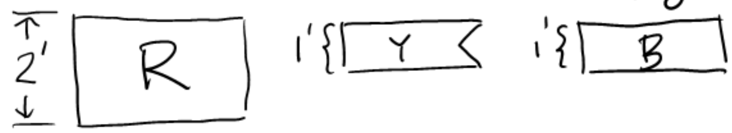
$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0$$

has distinct roots $q_1\ q_2\ q_3 \cdots q_k$. Then

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n \qquad *$$

is a solution of the recurrence relation for any constants c_i. Moreover, given any initial values $h_0\ h_1\ h_2 \cdots h_k$ values of c_i can be found so so that $*$ produces these initial values.

Example An unlimited supply of the following flags is available.



(red, yellow and blue)

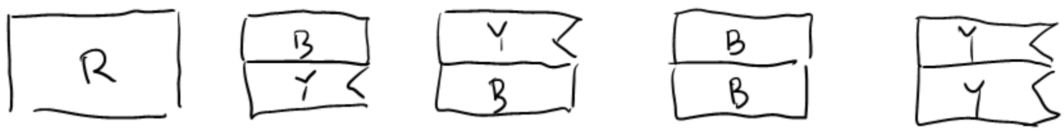
In how many ways can these be arranged on a n-foot flagpole?

Let $h_n = (\#\text{ of ways for an } n\text{-foot pole})$

$h_0 = 1$ (no flags at all)

$h_1 = 2$

$h_2 = 5$

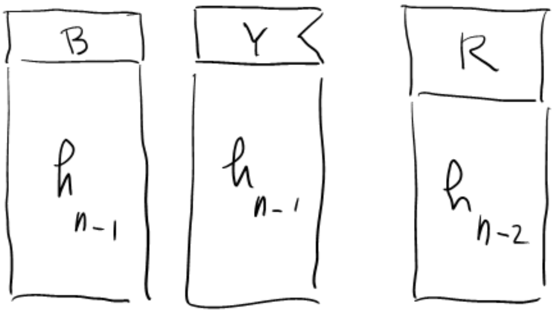


$h_3 = 12$

$2^3 = 8$

Recurrence relation is

$$h_n = \underbrace{h_{n-1}}_{\text{blue on top}} + \underbrace{h_{n-1}}_{\text{yellow on top}} + \underbrace{h_{n-2}}_{\text{red on top}}$$



i.e $h_n = 2h_{n-1} + h_{n-2}$ $\rightsquigarrow$ $h_n - 2h_{n-1} - h_{n-2}$

Characteristic equation: $x^2 - 2x - 1 = 0$

Roots $\dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2\cdot 1} = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

General solution $h_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$

Now find c_1 and c_2

$(n=0)$ $c_1(1+\sqrt{2})^0 + c_2(1-\sqrt{2})^0 = 1$

$(n=1)$ $c_1(1+\sqrt{2})^1 + c_2(1-\sqrt{2})^0 = 2$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1(1+\sqrt{2}) + c_2(1-\sqrt{2}) = 2 \end{cases} \longrightarrow c_2 = 1 - c_1$$

(now plug into second equation)

$$c_1(1+\sqrt{2}) + (1-c_1)(1-\sqrt{2}) = 2$$

$$c_1 + c_1\sqrt{2} + 1 - \sqrt{2} - c_1 + c_1\sqrt{2} = 2$$

$$2c_1\sqrt{2} = 1 + \sqrt{2}$$

$$c_1 = \frac{1+\sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2}+2}{4} = \boxed{\frac{2+\sqrt{2}}{4}}$$

$$c_2 = 1 - c_1 = \frac{4}{4} - \frac{2+\sqrt{2}}{4} = \boxed{\frac{2-\sqrt{2}}{4}}$$

<u>Answer</u> $\boxed{h_n = \dfrac{2+\sqrt{2}}{4}(1+\sqrt{2})^n + \dfrac{2-\sqrt{2}}{4}(1-\sqrt{2})^n}$

Theorem 7.2.1 requires that the roots of the characteristic equation be distinct. Let's look at what happens when they're not.

What to do if characteristic equation has repeated roots

Motivational Example

Solve $h_n = 4h_{n-1} - 4h_{n-2}$ with initial values $1, -3, -16, \ldots$

$$h_n - 4h_{n-1} + 4h_{n-2} = 0$$

$$x^n - 4x^{n-1} + 4x^{n-2} = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)(x-2) = 0$$

Characteristic equation has repeated roots 2, 2. Theorem 7.2.1 does not apply, but let's try it anyway

$$h_n = c_1 2^n + c_2 2^n = c2^n$$

$$(n=0) \quad h_0 = c2^0 = 1 \Rightarrow c=1 \Rightarrow \boxed{h_n = 2^n}$$

Problem! Does not match initial values!

Idea:

2 is a root

$$f(x) = x^n - 4x^{n-1} + 4x^{n-2} = x^{n-2}(x-2)^2$$

$$f'(x) = nx^{n-1} - 4(n-1)x^{n-2} + 4(n-2)x^{n-3} = (n-2)x^{n-3}(x-2)^2 + x^{n-2}2(x-2)$$

$$xf'(x) = nx^n - 4(n-1)x^{n-1} + 4(n-2)x^{n-2} = (n-2)x^{n-2}(x-2)^2 + x^{n-1}2(x-2)$$

Now $\underbrace{n2^n}_{h_n} - 4\underbrace{(n-1)2^{n-1}}_{h_{n-1}} + 4\underbrace{(n-2)2^{n-2}}_{h_{n-2}} = 0$

Let $h_n = n2^n$. This becomes $\boxed{h_n - 4h_{n-1} + 4h_{n-2} = 0}$

In other words, $h_n = n2^n$ is a solution to the recurrence relation, as is $h_n = 2^n$. Now let's combine these.

$$h_n = c_1 2^n + c_2 n 2^n \qquad \begin{cases} (n=0) & h_0 = c_1 2^0 + c_2 0 2^0 = 1 \\ (n=1) & h_1 = c_1 2^1 + c_2 \cdot 1 \cdot 2 = -3 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -\frac{5}{2} \end{cases}$$

Solution $\boxed{h_n = 2^n - \frac{5}{2} n 2^n}$

Check
$$h_0 = 2^0 - \frac{5}{2} 0 \cdot 2^0 = 1$$
$$h_1 = 2^1 - \frac{5}{2} 1 \cdot 2^1 = -3$$
$$h_2 = 2^2 - \frac{5}{2} 2 \cdot 2^2 = -16$$

If there were more than 2 repeated roots we could use the same trick, with higher derivatives. This leads to our next theorem

Theorem 7.2.2 Suppose the characteristic equation of a recurrence relation has roots $q_1\ q_2 \cdots q_t$, and and for each $1 \le i \le t$, q_i has multiplicity s_i. Then the general solution of the recurrence relation is

$$
\begin{aligned}
h_n = {} & c_{11} q_1^n + c_{12} n q_1^n + c_{13} n^2 q_1^n + \cdots + c_{1s_1} n^{s_1 - 1} q^n \\
& + c_{21} q_2^n + c_{22} n q_2^n + c_{23} n^2 q_2^n + \cdots + c_{2s_2} n^{s_2 - 1} q^n \\
& \vdots \\
& + c_{t1} q_t^n + c_{t2} n q_t^n + c_{t3} n^2 q_t^n + \cdots + c_{ts_t} n^{s_t - 1} q^n
\end{aligned}
$$

Example Solve $h_n = -4h_{n-1} - 6h_{n-2} - 4h_{n-3} - h_{n-4}$ with initial values $1, -4, 15, -40$.

$$h_n + 4h_{n-1} + 6h_{n-2} + 4h_{n-3} + h_{n-4} = 0$$

$$x^n + 4x^{n-1} + 6x^{n-2} + 4x^{n-3} + x^{n-4} = 0$$

$$x^4 + 4x^3 + 6x^2 + 4x + 1 = 0 \leftarrow \text{Characteristic Equation}$$

$$(x+1)^4 = 0$$

$$(x+1)(x+1)(x+1)(x+1) = 0$$

Roots are $-1, -1, -1, -1$.

General solution $h_n = c_1(-1)^n + c_2 n(-1)^n + c_2 n^2(-1)^n + c_3 n^3(-1)^n$

$$
\left.
\begin{array}{lll}
(n=0) & c_1 & = 1 \\
(n=1) & -c_1 - c_2 - c_3 - c_4 & = -4 \\
(n=2) & c_1 + 2c_1 + 4c_3 + 8c_4 & = 15 \\
(n=3) & -c_1 - 3c_2 - 9c_3 - 27c_4 & = -40
\end{array}
\right\}
\begin{array}{l}
c_1 = 1 \\
c_2 = 1 \\
c_3 = 1 \\
c_4 = 1
\end{array}
$$

Answer: $\boxed{h_n = (-1)^n + n(-1)^n + n^2(-1)^n + n^3(-1)^n}$