Section 6.5 Another Forbidden Position Problem

This is really just another exercise in inclusion-exclusion.

We are interested in permutations of $S = \{1, 2, 3, \cdots n\}$ for which <u>none</u> of the patterns $12, 23, 34, 45, \cdots, (n-1)n$ occur.

For example, when $n = 3$ we have the following permutations:

123 213 312
132 231 321 ← we are interested in these.

<u>Definition</u> Q_n is the number of permutations of $S = \{1, 2, 3, \ldots, n\}$ for which <u>none</u> of the patterns $12, 23, 34, \cdots (n-1)n$ occur.

$Q_1 = 1 \qquad Q_2 = 1 \qquad Q_3 = 3 \qquad Q_4 = 11$ (as in text.)

<u>Goal</u>: Find a formula for Q_n.

Strategy. Let
$\mathcal{U}$ be the set of <u>all</u> permutations of S, so $|\mathcal{U}| = n!$

$A_1 \subseteq \mathcal{U}$ be permutations in which 12 occurs

$A_2 \subseteq \mathcal{U}$ " " " " " " " 23 " " "

$A_3 \subseteq \mathcal{U}$ " " " " " " " 34 " " "

$\vdots$

$A_{n-1} \subseteq \mathcal{U}$ " " " " " " $(n-1)n$ " "

We are interested in the permutations $\overline{A_1 \cup A_2 \cup \cdots \cup A_{n-1}}$
$= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}}$

By inclusion-exclusion, $Q_n = |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n-1}}|$

$n! - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \cdots$

To work this out we need to find the cardinalities of all the intersections

Regard A_1 as the set of permutations of the $n-1$ symbols

$$\underline{12}, 3, 4, 5, \dots n.$$

Thus $|A_1| = (n-1)!$ and similarly $\boxed{|A_i| = (n-1)!}$

Now consider $A_i \cap A_j$. These intersections fall into two categories, which we explain by example.

First, consider the case when i & j are consecutive. Any permutation in $A_1 \cap A_2$ has the pattern 123, so look at it as a permutation of $\underline{123}, 4, 5, 6, \dots n$. Then $|A_1 \cap A_2| = (n-2)!$

Next suppose i & j are not consecutive, like in $A_1 \cap A_3$. A permutation in this set contains <u>both</u> patterns 12 and 34, so it is a permutation of $\underline{12}, \underline{34}, 5, 6, 7, 8 \dots n$. Again we have $|A_i \cap A_j| = (n-2)!$

Conclusion $\boxed{|A_i \cap A_j| = (n-2)!}$

Now, the case $A_i \cap A_j \cap A_k$ breaks into several categories

$A_1 \cap A_2 \cap A_3 \leftarrow$ permutations of $\underline{1234}, 5, 6, \dots, n$, so $(n-3)!$ of these

$A_1 \cap A_2 \cap A_4 \leftarrow$ permutations of $\underline{123}, \underline{45}, 6, 7, \dots n$, so $(n-3)!$ of these

$A_1 \cap A_3 \cap A_5 \leftarrow$ permutations of $\underline{12}, \underline{34}, \underline{56}, 7, 8, \dots n$, so $(n-3)!$ of these

Conclusion $\boxed{|A_i \cap A_j \cap A_k| = (n-3)!}$

Continuing, the intersection of any m of the A_i has cardinality $(n-m)!$. By inclusion-exclusion,

$$\boxed{Q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \dots \pm \binom{n-1}{n-1}1!}$$

<u>Ex</u> $Q_4 = 4! - \binom{3}{1}3! + \binom{3}{2}2! - \binom{3}{3}1! = 24 - 18 + 6 - 1 = 11$

$Q_5 = 5! - \binom{4}{1}4! + \binom{4}{2}3! - \binom{4}{3}2! + \binom{4}{4}! = 53$

<u>Exercise</u> Show $Q_n = D_n + D_{n-1}$