Chapter 5 The Binomial Coefficients

§5.1 Pascal's Formula

The following arrangement is called Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} \\
 & & & & & & \vdots & & & & & &
\end{array}
\longrightarrow
\begin{array}{cccccccccccccc}
 & & & & & & 1 & & & & & & & \leftarrow \text{row } 0 \\
 & & & & & 1 & & 1 & & & & & & \leftarrow \text{row } 1 \\
 & & & & 1 & & 2 & & 1 & & & & & \leftarrow \text{row } 2 \\
 & & & 1 & & 3 & & 3 & & 1 & & & & \leftarrow \text{row } 3 \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & \leftarrow \text{row } 4 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \leftarrow \text{row } 5 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \leftarrow \text{row } 6 \\
 & & & & & & \vdots & & & & & & &
\end{array}
$$

The numbers in this triangle are called binomial coefficients

This triangle reflects all of the identities that we have seen so far

For example $\binom{n}{k} = \binom{n}{n-k}$ is reflected in the symmetry across the vertical axis.

Also $\binom{n}{0} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n$, is reflected in the fact that the entries in the n^{th} row add up to 2^n.

Another fundamental relationship between the binomal coefficients is

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Number of k-element subsets of $\{0, 1, 2, \ldots n\}$

Number of such k-element subsets that contain 0. Select 0 plus k-1 elements from $\{1, 2, 3, \ldots n\}$

Number of such k-element subsets not containing 0. Select k elements from $\{1, 2, 3 \ldots n\}$

In Pascal's triangle this means any entry in the interior of the triangle is the sum of the two entries above it

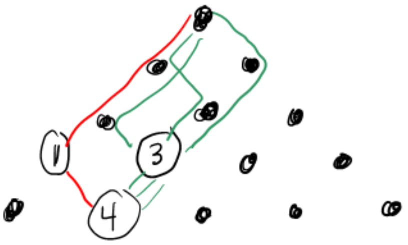
$$
\begin{array}{ccc}
\binom{n}{k-1} & & \binom{n}{k} \\
 & \searrow \quad \swarrow & \\
 & \binom{n+1}{k} &
\end{array}
$$

Theorem 5.1.1 (Pascal's Formula) If $1 < k < n$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

There are many interesting patterns in Pascal's Triangle. One of them is this:



Any entry of Pascal's Triangle equals the number of different routes from the apex to it, where in any step we are allowed to move only down right ↘ and down left ↙

Example

1 route to 1

3 routes to 3

1+3=4 routes to 4

Section 5.2 The Binomial Theorem

A two-term expression such as $x+y$ is called a binomial. We next discuss the Binomal Theorem, which tells the coefficients of $(x+y)^n$

It is a fundamental fact that the coefficients of $(x+y)^n$ are the entries from row n of Pascal's Triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$(x+y)^0 = \cdots\cdots\cdots\cdots \mathbf{1}$$
$$(x+y)^1 = \cdots\cdots\cdots \mathbf{1}x + \mathbf{1}y$$
$$(x+y)^2 = \cdots\cdots \mathbf{1}x^2 + \mathbf{2}xy + \mathbf{1}y^2$$
$$(x+y)^3 = \cdots\cdots \mathbf{1}x^3 + \mathbf{3}x^2y + \mathbf{3}xy^2 + \mathbf{1}y^3$$
$$(x+y)^4 = \cdots \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4$$
$$(x+y)^5 = \mathbf{1}x^5 + \mathbf{5}x^4y + \mathbf{10}x^3y^2 + \mathbf{10}x^2y^3 + \mathbf{5}xy^3 + \mathbf{1}y^5$$

Theorem 5.2.1 (Binomial Theorem)

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n}x^0 y^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

One way to see this is to "formally" multiply the binomial out:

$$(x+y)^3 = (x+y)(x+y)(x+y) = \underbrace{xxx}_{\binom{3}{0}\text{ ways to get 0 y's}} + \underbrace{xxy + xyx + yxx}_{\binom{3}{1}\text{ ways to get 1 y}}$$
$$+ \underbrace{xyy + yxy + yyx}_{\binom{3}{2}\text{ ways to get 2 y's}} + \underbrace{yyy}_{\binom{3}{3}\text{ ways to get 3 y's}}$$

$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

There are several proofs of The Binomial Theorem in the text. Read them.

The Binomial Theorem is useful when you have to expand powers of binomial expressions.

Ex $(2a-b)^4 = \left((2a)+(-b)\right)^4$

$$= (2a)^4 + \binom{4}{1}(2a)^3(-b) + \binom{4}{2}(2a)^2(-b)^2 + \binom{4}{3}(2a)(-b)^3 + (-b)^4$$

$$= 16a^4 - 4\cdot 8a^3b + 6\cdot 4a^2b^2 - 4\cdot 2ab^3 + b^4$$

$$= 16a^4 - 32a^2b + 24a^2b^2 - 8ab^3 + b^4$$

The Binomial Theorem also leads to some identities - some old and some new.

Ex $2^n = (1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}\cdot 1 + \binom{n}{2}1^{n-2}1^2 + \cdots + \binom{n}{n}1^0 1^n$

i.e. $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

Setting $y=1$ in the Binomial Theorem yields

$$\boxed{\text{Theorem 5.2.2} \quad (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=0}^{n}\binom{n}{n-k}x^k}$$

Ex $1\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + 2^3\binom{n}{3} + \cdots 2^n\binom{n}{n}$

$$= \sum_{k=0}^{n}\binom{n}{k}2^k = (1+2)^n = 3^n$$

Spot check $\binom{4}{0} + 2\binom{4}{1} + 4\binom{4}{2} + 8\binom{4}{3} + 16\binom{4}{4}$

$$= 1 + 2\cdot 4 + 4\cdot 6 + 8\cdot 4 + 16$$

$$= 1 + 8 + 24 + 32 + 16 = 81 = 3^4$$

Read § 5.4 on Unimodality of Binomial Coefficients. It just proves That the sequence

$$\binom{n}{0}\ \binom{n}{1}\ \binom{n}{2}\ \binom{n}{3} \cdots \binom{n}{n}$$

increases from $\binom{n}{0}$ to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ and Then decreases from $\binom{n}{\lceil \frac{n}{2} \rceil}$ to $\binom{n}{n}$. This may seem obvious. Indeed the proof is not hard.

§ 5.3 Identities

We here take note of more identities involving Binomial Coefficients. Some of these will be useful for us later. We already know:

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

Another noteworthy identity is

$$\boxed{k\binom{n}{k} = n\binom{n-1}{k-1}}$$

Reason: This is saying

$$k\frac{n!}{k!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!}$$

and this holds by cancellation

Here is another.

$$\boxed{\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots (-1)^n\binom{n}{n} = 0}$$

Reason Binomial Theo 5.1.3

$$0 = (1-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k$$

From this we get:

$$\boxed{\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots}$$

Other identities can be obtained from differentiating the Binomial Formula

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$\frac{d}{dx}\left[(1+x)^n\right] = \frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}x^k\right]$$

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}$$

Now plug in $x=1$ to get

$$n(1+1)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}\cdot 1^{k-1}$$

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

Thus we have

$$\boxed{0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}}$$

Here is another interesting identity.

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}}$$

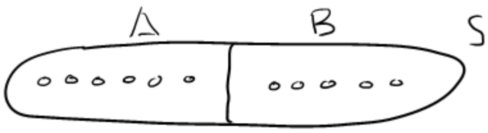
This can be proved with induction, but we will here take an altertanitive approach. The technique is called <u>combinatorial proof</u>. The idea is to count the same thing in different ways, so the expressions you get from the two ways must be equal.

<u>Theorem</u> $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

<u>Proof</u> Let S be a set with $2n$ elements. Lets count the number of n-element subsets of S (i.e. The n-combinations of S).

On one hand there are $\binom{2n}{n}$ n-combinations of S.

Now we count the n-combinations in a different way. Divide S into two pieces, so $S = A \cup B$ with $|A| = |B| = n$.



If $0 \le k \le n$, then we can make an n-combination of S by choosing k things in A, then $n-k$ things in B. Put them together and get an n-combination of S.
By the multiplication principle, the number of ways to do this is

$$\binom{n}{k}\binom{n}{n-k}$$

Thus the total number of n-combinations of S is

$$\underbrace{\binom{n}{0}\binom{n}{n-0}}_{\substack{0 \text{ things in } A \\ n \text{ things in } B}} + \underbrace{\binom{n}{1}\binom{n}{n-1}}_{\substack{1 \text{ things in } A \\ n-1 \text{ things in } B}} + \underbrace{\binom{n}{2}\binom{n}{n-2}}_{\substack{2 \text{ things in } A \\ n-2 \text{ things in } B}} + \cdots \underbrace{\binom{n}{n}\binom{n}{n-n}}_{\substack{n \text{ things in } A \\ 0 \text{ things in } B}}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Now were counted the number of n-combinations of S in two ways. First way gave $\binom{2n}{n}$. Second way gave $\sum_{k=0}^{n} \binom{n}{k}^2$. Thus $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$. $\blacksquare$