Section 3.3 Combinations of Sets

Definitions Suppose S is a finite set.
A r-permutation of S is an r-element list made from elements of S
A r-combination of S is an r-element set made from elements of S
(ie., an r-element subset of S)

Example $S = \{a, b, c, d\}$

3-permutations of S abc acb abd adb etc.
3-combinations of S $\{a,b,c\}$ $\{a,b,d\}$ etc.

Definitions Suppose S is a set with $|S| = n$.
The number of r-permutations of S is denoted $P(n,r)$.
The number of r-combinations of S is denoted $\binom{n}{r}$

Recall $P(n,r) = \frac{n!}{(n-r)!}$

Example $S = \{a, b, c, d\}$ so $|S| = 4$

r	r-combinations of S	$\binom{4}{r}$
-2	(none)	0
-1	(none)	0
0	$\{\ \}$	1
1	$\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$	4
2	$\{a,b\}$ $\{a,c\}$ $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$	6
3	$\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$	4
4	$\{a,b,c,d\}$	1
5	(none)	0

Thus
$\binom{4}{0} = 1$ $\binom{4}{1} = 4$
$\binom{4}{2} = 6$ $\binom{4}{3} = 4$
$\binom{4}{4} = 1$

We have a formula for $P(n,r)$. Now we will derive one for $\binom{n}{r}$.
To understand how to get it consider the special case $n=4$ and $r=3$.
Make a table with columns headed by the $\binom{4}{3} = 4$ 3-combinations of $S = \{a,b,c,d\}$
Below each r-combination, list all its permutations. You get all 3-permutations

$\{a,b,c\}$	$\{a,b,d\}$	$\{a,c,d\}$	$\{b,c,d\}$
abc	abd	acd	bcd
acb	adb	adc	bdc
bac	bad	cad	cbd
bca	bda	cda	cdb
cab	dab	dac	dbc
cba	dba	dca	dcb

$\uparrow$ 3! $\downarrow$

$\longleftarrow \binom{4}{3} \longrightarrow$

Box contains all 3-permutations of 4-element set S. Thus

$$3!\binom{4}{3} = P(4,3).$$

Thus $\binom{4}{3} = \frac{1}{3!}P(4,3) = \frac{4!}{3!(4-3)!}$

By the same reasoning $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Formula If $0 \leq r \leq n$ then $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Otherwise $\binom{n}{r} = 0$.

Example How many 3-combinations of $S = \{a,b,c,d,e\}$?

Answer: $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1} = 5\cdot2 = \boxed{10}$

Inded the 10 3-combinations are $\{a,b,c\}$ $\{a,b,d\}$ $\{a,b,e\}$ $\{a,c,d\}$ $\{a,c,e\}$ $\{a,d,e\}$ $\{b,c,d\}$ $\{b,c,e\}$ $\{b,d,e\}$, $\{c\,d\,e\}$

Example How many possible different 5-card hands can be dealt off of a standard 52-card deck?

Solution: Think of a 5-card hand as a 5-combination of the set of 52 cards. Answer is $\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot47!}{5\cdot4\cdot3\cdot2\cdot1\cdot47!}$

$= \frac{52\cdot51\cdot\overset{5}{\cancel{50}}\cdot49\cdot\overset{4}{\cancel{48}}}{\cancel{10}\cdot\cancel{12}} = 52\cdot51\cdot5\cdot49\cdot4 = \boxed{2,598,960}$

Example How many 5-card hands can be dealt off a 52-card deck in which 2 cards are clubs and 3 cards are hearts?

Answer $\binom{13}{2}\binom{13}{3} = \frac{13!}{2!\,11!}\,\frac{13!}{3!\,10!} = \frac{13\cdot12}{2}\,\frac{13\cdot12\cdot11}{3\cdot2} = 13\cdot6\cdot13\cdot2\cdot11 = 22308$

chose 2 of 13 clubs

chose 3 of 13 hearts

Example How many 5-card hands are there for which two cards are of one suit and three are of another

Answer $4\binom{13}{2}\cdot3\binom{13}{3} = 4\cdot\frac{13\cdot12}{2}\cdot3\,\frac{13\cdot12\cdot11}{6} = 803088$

4 ways to choose the suit for 2 cards

ways to choose 2 cards of that suit

Now 3 ways to choose suit for the 3 cards

ways to choos 3 cards of That suit

Example How many 10-letter strings (repetition allowed) can be made from A, B, C, D, E, F are there that contain exactly 3 A's?

Solution: There are $\binom{10}{3}$ ways to choos slots for the 3 A's

_ A _ _ _ A A _ _ _

For any such choice, each of the remaining 7 slots can be filled in 5 ways (choice of B, C, D, E, F), i.e. in 5^7 ways

Answer $5^7\binom{10}{3} = 5^7\,\frac{10!}{3!(10-3)!} = \frac{5^7\,10\cdot9\cdot8}{6} = 5^7\cdot10\cdot3\cdot4$

$= \boxed{9,375,000}$

Example How many 7-card hands are there with 2 cards of one suit, 4 cards of another suit and 1 card of a 3rd suit?

(choose 1st suit)(select 2 cards of 1st suit)(chose 2nd suit)(select 4 cards of 2nd suit)(choose 3rd suit)(select 1 card of 3rd suit)

Ans: $4 \times \binom{13}{2} \times 3 \times \binom{13}{4} \times 2 \times 13$

$= 6,960,096$ such hands

Ex How many integer solutions of there of $x+y+z+w=100$, where $x,y,z,w \geq 0$?

Example $x=2$ $y=5$ $z=3$ $w=90$,

$x=1$, $y=0$, $z=2$, $w=97$, etc.

Solution: model solutions as follows; as a sequence of stars & bars

$x=2$ $y=5$ $z=3$ $w=90$

$$\underbrace{\ast\ast}_{2} \mid \underbrace{\ast\ast\ast\ast\ast}_{5} \mid \underbrace{\ast\ast\ast}_{3} \mid \underbrace{\ast\ast\ast\cdots\ast}_{90}$$

$x=1$ $y=0$ $z=2$ $w=97$

$$\underbrace{\ast}_{1} \mid \underset{\overset{\uparrow}{0}}{} \mid \underbrace{\ast\ast}_{2} \mid \underbrace{\ast\ast\ast\cdots\ast}_{97}$$

$x=100$, $y,z,w=0$

$$\underbrace{\ast\ast\ast\cdots\ast}_{100} \underset{\overset{\uparrow}{0}}{\mid} \underset{\overset{\uparrow}{0}}{\mid} \underset{\overset{\uparrow}{0}}{\mid}$$

Each solution corresponds to a sequence of 103 symbols, 100 stars and 3 bars. Any solution is encoded as

x stars | y stars | z stars | w stars

How many such sequences? To make one, start with 103 slots and select 3 of then for the bars.

Answer $\binom{103}{3} = \frac{103!}{3!(103-3)!} = \frac{103 \cdot 102 \cdot 101}{6} = 176,851$ solutions.

Example How many solutions to $x+y+z+w=100$ if $x \geq 10$, $y \geq 0$, $z \geq 0$ $w \geq -5$

Think of this as $\underbrace{(x-10)}_{\geq 0} + \underbrace{y}_{\geq 0} + \underbrace{z}_{\geq 0} + \underbrace{(w+5)}_{\geq 0} = 100-10+5$

We can model a solution with 95 stars and 3 bars

$$\underbrace{\ast\ast\ast\ast\cdots\ast}_{x-10} \mid \underbrace{\ast\ast\ast\cdots\ast}_{y} \mid \underbrace{\ast\ast\ast\cdots\ast}_{z} \mid \underbrace{\ast\ast\ast\cdots\ast}_{w+5}$$

Thus 98 slots all together, and choose 3 for the bars.

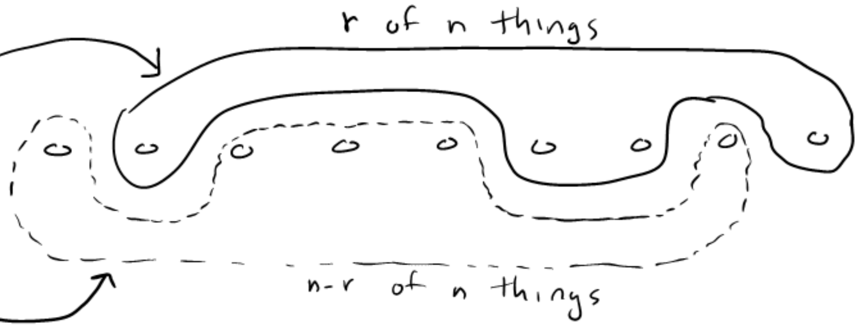
Answer: $\binom{98}{3} = \frac{98 \cdot 97 \cdot 96}{6}$

Now we are going to look at some identities involving the numbers $\binom{n}{r}$

$$\boxed{\text{FACT: } \binom{n}{r} = \binom{n}{n-r}}$$

Reason: For any choice of r out of n things...

... there is a corresponding choice of n-r out of n of the things



Alternatively you could note that

$$\begin{cases} \binom{n}{r} = \dfrac{n!}{r!(n-r)!} \\ \binom{n}{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!\,r!} \end{cases}$$

equal!

Also note

$$\boxed{\text{FACTS: } \binom{n}{0}=1 \quad \binom{n}{n}=1 \quad \binom{n}{1}=n \quad \binom{n}{n-1}=n}$$

For the next identity recall that if a set has n elements, then it has 2^n subsets (including ϕ).

Reason Consider making a subset $X \subseteq S$ where $|S| = n$
For each element of S you have 2 options - include the element in X, or not. Thus the number of ways to make X are $\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdots\cdots 2}_{n} = 2^n$

Example $S = \{a, b, c\}$
Then S has $2^3 = 8$ subsets: $\underbrace{\phi}_{\binom{3}{0}}$ $\underbrace{\{a\} \; \{b\} \; \{c\}}_{\binom{3}{1} \text{ of these}}$ $\underbrace{\{a,b\} \; \{a,c\} \; \{b,c\}}_{\binom{3}{2} \text{ of these}}$ $\underbrace{\{a,b,c\}}_{\binom{3}{3} \text{ of these}}$

$$\boxed{\text{FACT } \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n}$$

Reason.

Consider the situation of a set S with $|S| = n$.

# of 0-element subsets of S + # of 1-element subsets of S + # of 2-element subsets of S ... # of n-element subsets of S = Total number of subsets of S