Carsten Thomassen and I proved that in any graph $G$, the number of cycles containing a specified edge as well as all the odd-degree vertices is odd if and only if $G$ is eulerian. Where all vertices have even degree this is a theorem of Sunichi Toida and where all vertices have odd degree it is Andrew Thomason’s generalization of Smith’s Theorem.

Ken Berman extended Thomason’s Theorem to trees: he used a counting argument to prove that if $T$ is a spanning tree of a graph $G$ where all vertices in $G-E(T)$ have odd degree, there is an even number of spanning trees of $G$ with the same degree as $T$ at each vertex. I give a common generalization of these results to a parity theorem about (not necessarily spanning) trees.

Andrew Thomason proved his theorem by constructing an exchange graph $X(G)$, his lollipop graph, in which the odd-degree vertices correspond precisely to the things he wants to show there are an even number of, namely the hamiltonian cycles in $G$ containing the specified edge. This provides an algorithm for given one of the objects, finding another, by walking in $X(G)$ from one odd-degree vertex to another.

I have extended Thomason’s algorithm to one which, in a non-eulerian graph, finds a second cycle containing a specified edge and all the odd-degree vertices, and more generally finds a second tree with certain properties.

I will discuss some other parity theorems about paths, cycles, and trees in graphs and attempts to find exchange graph proofs of them.