The Heegaard Floer d-invariants and integral surgery

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Heegaard Floer homology

Heegaard Floer homology (Ozsváth and Szabó, 2001) is a package of 3- and 4-manifold invariants.

$$Y \rightsquigarrow \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(Y) \cong H^2(Y)} HF^\circ(Y, \mathfrak{t})$$

- Y is a closed, oriented 3-manifold (usually a $\mathbb{Q}HS^3$).
- $\circ \in \{+, -, \infty, \wedge\}$
- HF° is an $\mathbb{F}[U]$-module (or an $\mathbb{F}$ vector space).
- Take $+$:

$$HF^+(Y, \mathfrak{t}) \cong \underbrace{\mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U]}_{tower = \mathcal{T}^+} \oplus \underbrace{HF_{red}(Y, \mathfrak{t})}_{torsion}$$

Examples

1. $HF^+(S^3) = \mathcal{T}^+$
2. $HF^+(L(3,1)) = \mathcal{T}^+ \oplus \mathcal{T}^+ \oplus \mathcal{T}^+$
3. $HF^+(\Sigma(2,3,7)) = \mathcal{T}^+ + \mathbb{F}$

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An *L-space* is a rational homology sphere whith $HF_{red}(Y) = 0$.

Examples

Fact: $HF^+(Y)$ has an absolute $\mathbb{Q}$-grading.

1. $HF^+(S^3) = \mathcal{T}^+_{(0)}$
2. $HF^+(L(3,1)) = \mathcal{T}^+_{(1/2)} \oplus \mathcal{T}^+_{(-1/6)} \oplus \mathcal{T}^+_{(-1/6)}$
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Definition

$d := d(Y, \mathfrak{t})$ is the minimal grading of the tower $\mathcal{T}^+_{(d)}$

Applications of the d-invariants

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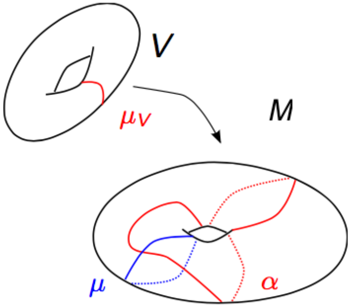
B More applications: lens space surgeries and band surgery

(to be described today)

Distance one lens space surgeries

Consider Dehn surgery along a knot K in the lens space $L(3,1)$.

- $M = L(3,1) - nbhd(K)$
- μ is a meridian of K.
- $M(\mu) = L(3,1)$
- $M(\alpha) =$ some other manifold.
  I'll want $L(n,1)$.



We say α and μ are *distance one* in ∂M because they intersect geometrically once.
(Also called an integral surgery.)

First application: lens space realization

Theorem (Lidman-Moore-Vazquez, 2018)

The lens space $L(n,1)$ is obtained by a distance one surgery along any knot in the lens space $L(3,1)$ if and only if

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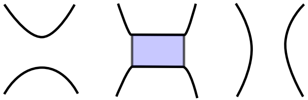
Remarks:

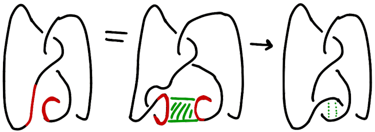
1. Partial generalization of the lens space realization problem in S^3 (compare Greene, 2013).
2. Cyclic Surgery Theorem (CGLS 1987):
   If $M = Y - K$ is not Seifert fibered, then any pair of cyclic surgery slopes are of distance at most one.

Second application: band surgery obstructions

Band surgery is a topological operation that relates a pair of knots or links.



Note that band surgery may be realized as the resolution of a crossing in some diagram (or vice versa).



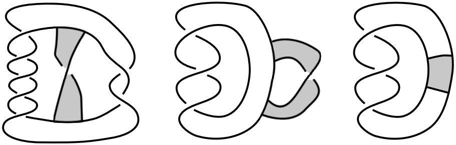
Band surgery along the trefoil

Theorem (Lidman-Moore-Vazquez, 2018)

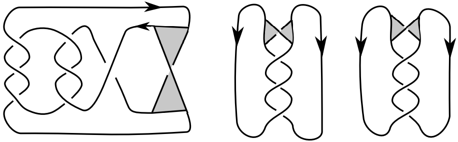
The torus knot or link $T(2,n)$ is obtained from $T(2,3)$ by a banding if and only if

$$n \in \{\pm 1, \pm 2, 3, 4, -6, 7\}.$$

Proof of Theorem 2 ($\Leftarrow$)



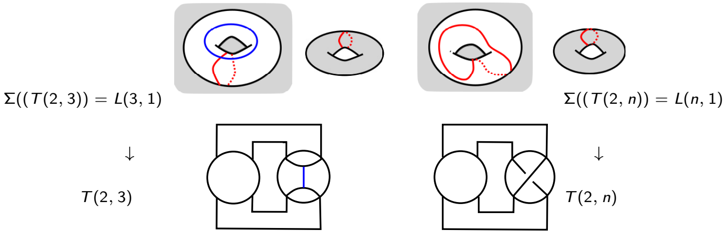
Bandings from $T(2,3)$ to $T(2,7)$, from $T(2,3)$ to itself, and from $T(2,3)$ to the unknot.



Bandings from $T(2,3)$ to $T(2,-6)$, $T(2,3)$ to $T(2,2)$, and $T(2,3)$ to $T(2,4)$.

Proof of Theorem 2 ($\Rightarrow$)

Theorem 2 follows from Theorem 1 after an application of "the Montesinos trick."



A band surgery from $T(2,3)$ to $T(2,n)$ lifts to an integral Dehn surgery (along some knot) in $L(3,1)$ to $L(n,1)$.

Idea of proof of Theorem 1

Theorem (Lidman-Moore-Vazquez, 2018)

The lens space $L(n,1)$ is obtained by a distance one surgery along any knot in the lens space $L(3,1)$ if and only if

$$n \in \{\pm 1, \pm 2, 3, 4, -6, 7\}.$$

General approach: Describe behavior of d-invariants under integral surgery in $L(3,1)$ and seek a contradiction when n is not on this list.

The d-invariants of manifolds related by surgery

Proposition (Ni-Wu, 2015)

Let $Y_p(K)$ denote p-surgery along a null-homologus knot K in an L-space Y. Then:

$$d(Y_p(K), \mathfrak{t}_i) = d(Y, \mathfrak{t}) + d(L(p,1), i) - 2N_{\mathfrak{t},i}.$$

Issue: what if K is nontrivial in homology?

We do Ni-Wu for homologically nontrivial knots in $L(3,1)$

Let $Y = L(3,1)$ and assume $Y' = L(3k \pm 1, 1)$ is obtained from an integral surgery along K in Y.

Proposition (Lidman-Moore-Vazquez, 2018)

We get similar formulas:

$$d(Y', \mathfrak{t}) = d(L(3k \pm 1, 3), 1) - 2N_0.$$

$$d(Y', \mathfrak{t} + PD[\mu]) = d(L(3k \pm -1, 3), 4) - 2N_1.$$

Where N_0 and N_1 are certain integer-valued knot invariants coming from the knot Floer complex.

We then use these formulas to analyze when $L(n,1)$ is obtained by surgery along any K in $L(3,1)$.

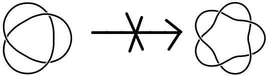
Example: $L(3,1)$ to $L(5,1)$

Here $3k-1=5$, so $k=2$. We have

$$\underbrace{d(L(5,1),0)}_{=-1} = \underbrace{d(L(5,3),1)}_{=0} - 2N_0$$

Contradiction: N_0 must be an integer.

Therefore no distance one surgery from $L(3,1)$ to $L(5,1)$ and consequently no band surgery



Mapping cone formula

Ozsváth and Szabó, 2008: mapping cone for knots
Ozsváth-Manolescu, 2010: mapping cone for links

$K \rightsquigarrow CFK^\infty(S^3, K)$; bifiltered complex.
(Using "minus") take the A_i^0 and A_i^1 subcomplexes.

$$CF^-(Y_N(K), \mathfrak{t}_i) \simeq A_i^0 \xrightarrow{v} A_i^1 \simeq CF^-(Y)$$

$$CF^-(Y_N(K), \mathfrak{t}_i) \simeq A_i^0 \xrightarrow{h} A_{i+p}^1 \simeq CF^-(Y)$$

The maps v and h give rise to the integer-valued knot invariants V_i and H_i:

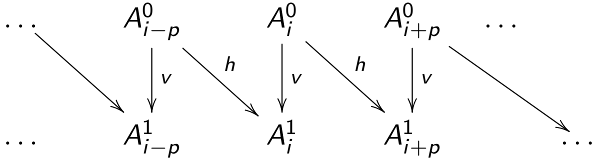
$$U^{V_i} : \mathcal{T}_i \to \mathcal{T}_i$$

$$U^{H_i} : \mathcal{T}_i \to \mathcal{T}_{i+p}$$

that eventually appear in all of the d-invariant formulas.

Mapping cone formula

Then assemble into a mapping cone complex.

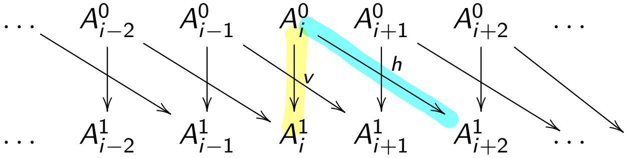


Theorem (Ozsváth and Szabó)

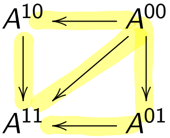
$$H_* \left(\textit{cone of } (v+h)\right) \cong HF^-(Y_p(K))$$

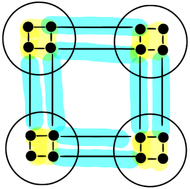
The mapping cone is quite complicated for links

For a knot K:



For a two-component link $L = L_1 \cup L_2$:





d-invariant formulas for linking number zero links

Theorem (Gorsky-Liu-Moore, 2018)

The d-invariants of integral surgeries on a two-component L–space link with linking number zero can be computed as follows:
(a) If $p_1, p_2 < 0$ then

$$d(S^3_{p_1,p_2}(L), (i_1, i_2)) = d(L(p_1, 1), i_1) + d(L(p_2, 1), i_2).$$

(b) If $p_1, p_2 > 0$ then

$$d(S^3_{p_1,p_2}(L), (i_1, i_2)) = d(L(p_1, 1), i_1) + d(L(p_2, 1), i_2) - 2N$$

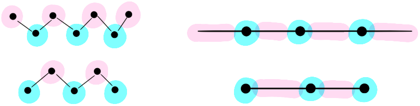
where N is a certain non-negative integer.
(c) If $p_1 > 0$ and $p_2 < 0$ then

$$d(S^3_{p_1,p_2}(L), (i_1, i_2)) = d(S^3_{p_1}(L_1), i_1) + d(L(p_2, 1), i_2).$$

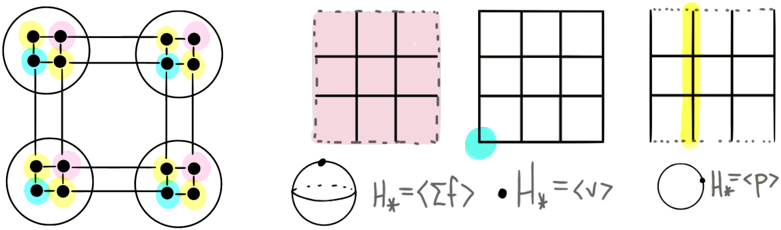
Proof idea

Replace mapping cone complex with an actual cell complex.

For knots:



For two-component links:



More fun with the d-invariants

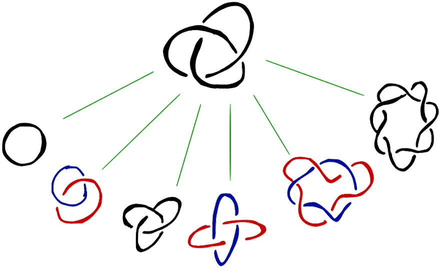
Theorem (Gorsky-Liu-Moore)

Let $L = L_1 \cup L_2$ have linking number zero.

1. *If L_i are unknots and L is an L-space link, then $S^3_{p_1,p_2}(L)$ is an L-space if and only if $p_i > 2\tau(L'_i - 2)$.*
2. *We describe Sato-Levine invariant of L and Casson invariant of $(\pm 1, \pm 1)$ surgery along L in terms of the h-function.*

Let $L = L_1 \cup \cdots \cup L_n$ have all linking numbers all zero. We use the d-invariants of surgery to:

3. *give bounds on the smooth four genus of links,*
4. *define concordance invariants, and*
5. *determine skein inequalities.*



Thank you for listening!