ADJACENCY OF THREE-MANIFOLDS AND BRUNNIAN LINKS

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Abstract. We introduce the notion of adjacency in three-manifolds. A three-manifold $Y$ is $n$-adjacent to another three-manifold $Z$ if there exists an $n$-component link in $Y$ and surgery slopes for that link such that performing Dehn surgery along any nonempty sublink yields $Z$. We characterize adjacencies from three-manifolds to the three-sphere, providing an analogy to Askitas and Kalfagianni’s results on $n$-adjacency in knots.

1. Introduction

A knot $K$ is said to be $n$-adjacent to a knot $K'$ if there exists a diagram of $K$ containing a set of $n$ crossings such that changing any nonempty subset of them yields a diagram of $K'$. Any knot that is adjacent to the unknot is, of course, unknotting number one, but the condition is much more restrictive. For example, nontrivial knots which are $n$-adjacent to the unknot for $n \geq 3$ have trivial Alexander polynomial, are non-fibered, non-alternating, and have vanishing Vassiliev invariants of degree less than $2n - 1$. These restrictions on their invariants are shown by Askitas-Kalfagianni [AK02] to result from a diagrammatic characterization of $n$-adjacent knots, $n \geq 3$, as those constructed from certain spatial chord diagrams called Brunnian-Suzuki graphs [AK02, Theorem 4.4]. A Suzuki graph is given by a collection of weighted, embedded arcs along an unknotted circle in $S^3$, where the arcs describe a pattern along which a sequence of bandings and clasps converts the graph into a knot.

In this article, we generalize the concept of $n$-adjacency from knots to three-manifolds. We say that a closed, oriented three-manifold $Y$ is \textit{integralely $n$-adjacent} to another three-manifold $Z$ if there exists an $n$-component link $L$ and integral multi-slope $p$ of the link such that performing Dehn surgery along any nonempty subset of $L$ yields $Z$. The triple realizing the adjacency will be denoted $(Y, L, p)$. Rational adjacency, denoted $(Y, L, \alpha)$, is defined similarly where rational surgeries are permitted. In an analogy to Askitas-Kalfagianni, we characterize all $n$-adjacencies to the three-sphere as those arising from particular Dehn surgeries along Brunnian-like links. This recovers Askitas-Kalfagianni’s diagrammatic characterization of knots adjacent to the unknot, and we similarly obtain a statement on the finite type invariants of three-manifolds.

Given a link in a three-manifold, and a choice of surgery slopes, the core curves of the surgery solid tori produce a new link in the surgered manifold. We call this the \textit{core} or \textit{dual} of the surgery. Using dual links, it is in fact quite easy to construct examples of adjacent manifolds for all $n$: Let $J$ be a Brunnian link in $S^3$, and perform $(\pm 1, \ldots, \pm 1)$-surgery on $J$. This yields a homology sphere $Y$ that is $n$-adjacent to the three-sphere via...
The Poincaré homology sphere results from \((1, 1, 1)\)-surgery on the Borromean rings. The meridians with surgery slopes \((0, 0, 0)\) realize a 3-adjacency of the Poincaré homology sphere to \(S^3\).

This three-component Hopf-Brunnian link \(J\) admits a \((1/2, 1, 1/2)\)-rational surgery to \(-L(3, 1)\). The core of \(J\) after surgery is a link \(L\) realizing the 3-adjacency of \(-L(3, 1)\) to \(S^3\). The homology sphere will be distinct from \(S^3\) provided that \(J\) is a non-trivial link. For a concrete example, \((1, 1, 1)\)-surgery on the Borromean rings produces the Poincaré homology sphere, which is 3-adjacent to \(S^3\), as shown in Figure 1. In fact, we will see shortly that all integral \(n\)-adjacencies for \(n \geq 3\) arise from surgery on Brunnian links.

Next, consider the example in Figure 2. An exercise in Kirby calculus shows that \((1/2, 1, 1/2)\)-surgery yields \(-L(3, 1)\) and induced surgery on any proper sublink yields \(S^3\). This surgery description will demonstrate that \(-L(3, 1)\) is 3-adjacent to \(S^3\). Thus, Hopf links can and do appear in the characterization of rational adjacencies. We define the family of Hopf-Brunnian links as follows. An \(n\)-component link is Hopf-Brunnian if all \(n - 1\) component sublinks are split unions of Hopf links and unknots. We prove:

**Theorem 1.1.** The triple \((Y, L, \alpha)\) realizes an \(n\)-adjacency to \(S^3\) with surgery core \(J\) if and only if \(J\) is Hopf-Brunnian, the dual surgery slopes of \(J\) are of the form \(1/k_i, k_i \in \mathbb{Z}^*\), and any proper Hopf sublink of \(J\) must have surgery slopes \(\pm(1, 1/2)\) or \(\pm(1/2, 1)\).

**Corollary 1.2.** The triple \((Y, L, p)\) realizes an integral \(n\)-adjacency to \(S^3\) with surgery core \(J\) if and only if \(J\) is Brunnian and the dual surgery slopes of \(J\) are \(\pm 1\) (signs need not be consistent).

**Corollary 1.3.** If \(Y\) is integrally \(n\)-adjacent to \(S^3\) for \(n \geq 3\), or rationally \(n\)-adjacent to \(S^3\) for \(n \geq 4\), then \(Y\) is an integer homology sphere.
Note that in the case \( n = 2 \), any \((1/k_1, 1/k_2), k_i \in \mathbb{Z}^*\), surgery along any link \( J_1 \cup J_2 \) of unknotted components in \( S^3 \) will yield a manifold adjacent to the three-sphere. In order to prove Theorem 1.1, we give a stronger characterization for self-adjacencies from the three-sphere to itself.

**Proposition 3.2** The triple \((S^3, J, \alpha)\) realizes an \( n \)-adjacency to \( S^3 \) if and only if \( J \) itself is a split union of Hopf links and unknots, all slopes \( \alpha_i = 1/k_i \), where \( k_i \in \mathbb{Z}^* \), and the surgery slopes of Hopf components are either \( \pm(1, 1/2) \) or \( \pm(1/2, 1) \).

Let us return to \( n \)-adjacency in knots. Using Theorem 1.1, we are now able to recover Askitas-Kalfagianni’s characterization of knots which are \( n \)-adjacent to the unknot [AK02, Theorem 4.4].

**Theorem 1.4** (Askitas-Kalfagianni). Let \( K \) be \( n \)-adjacent to the unknot for \( n \geq 3 \). Then \( K \) is the realization of a Brunnian-Suzuki \( n \)-graph.
Consider now the union of the arcs $\gamma_1, \ldots, \gamma_n$ together with the unknot $U$. Again by the Montesinos trick [Mon75], an arc with weight $(w, z)$ in the terminology of [AK02, Section 3] realizes a surgery in the branched double cover with slope $w + z/2$. The theorem will follow from the following claim.

**Claim 1.** The graph $G = U \cup \bigcup_{i=1}^{n} \gamma_i$ is a Brunnian Suzuki $n$-graph.

**Proof of claim.** Notice that no pair of arcs $\gamma_i$ and $\gamma_j$ have endpoints interleaved along the unknot because they would lift to a link $J_i \cup J_j$ of nonzero linking number, a contradiction. This means that $G$ is admissible, in the terminology of [AK02, Definition 3.2]. For any proper subset $I \subset \{1, \ldots, n\}$ of components of $J_1 \cup \ldots \cup J_n$, the quotient under $\tau$ is a proper subset of the $\gamma$ arcs. Because each subset of $J$ is an unlink, and there is a unique strong inversion $\tau$ on the unlink [KT80], these descend to arcs embedded disjointly in the spanning disk for the unknot. Moreover, because the surgery slopes on each $J_i$ are $\pm 1/2$, these descend to weighted arcs of the form $(0, \pm 1)$. Thus, subgraphs of $G$ with $n - 1$ arcs are standard. The graph $G$ is therefore a Brunnian-Suzuki $n$-graph, as in [AK02, Definition 3.4].

Note that Theorem 1.4 is the key ingredient in the claimed vanishing of the Vassiliev invariants of a knot $n$-adjacent to the unknot, as mentioned above.

In analogy with the work of Askitas-Kalfagianni, we are also able to apply Theorem 1.1 more generally to prove a vanishing result for finite-type invariants of homology spheres. (For a quick survey on finite-type invariants, see [Lin98].)

**Corollary 1.5.** Let $Y$ be a homology sphere which is integrally $n$-adjacent to $S^3$. Then all finite-type invariants of order less than $2n - 4$ vanish.

**Proof.** By Corollary 1.2, $Y$ is obtained by surgery on an $n$-component Brunnian link. The required vanishing result is given by [Mei06, Theorem 1.1].

2. **Dehn surgery**

2.1. **The dual perspective.** Let $Y$ be a closed, oriented three-manifold, and let $L = L_1 \cup \ldots \cup L_n$ denote a link in $Y$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote a multi-slope on $L$. The notation $Y_\alpha(L)$ denotes the three-manifold obtained by performing $\alpha_i$ Dehn surgery along $L_i$ for all $i = 1, \ldots, n$.

**Definition 2.1.** Consider the triple $(Y, L, \alpha)$, where $Y$ is a closed, oriented 3-manifold, $L = L_1 \cup \ldots \cup L_n$ is a link in $Y$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-slope on $L$. Let $Z$ be a closed, oriented three-manifold. If $Y_\alpha(L) = Z$ for any nonempty subset $I$ of $\{1, \ldots, n\}$, then $(Y, L, \alpha)$ realizes an $n$-adjacency to $Z$. We say that $Y$ is integrally $n$-adjacent to $Z$ if the multi-slopes are integral and rationally adjacent to $Z$ otherwise.

Notice that $n$-adjacency is not a symmetric relation.
In order to circumvent the difficulty in describing surgeries in arbitrary three-manifolds, we take the following perspective.

Suppose that $L$ is an $n$-component link in a three-manifold $Y$ with a surgery to $S^3$ along the multi-slope $\alpha$. Let $J$ be the core of the surgery in $S^3$, and write $J = J_1 \cup \ldots \cup J_n$. Then, we will associate to $J$ rational numbers $r = (r_1, \ldots, r_n)$ which describe how to “undo” the surgery performed on each $J_i$. To be more precise, in the exterior of $L$, we have two slopes $\eta_i$ and $\alpha_i$ on the boundary torus coming from $L_i$: $\eta_i$ is the meridian of $L_i$ and Dehn filling along $\alpha_i$ corresponds to the non-trivial surgery we are going to do to get to $S^3$. Viewing $J$ as a link in $S^3$, we still can view these slopes in the boundary of a neighborhood of $J_i$. Express $\eta_i = p_i \mu_i + q_i \lambda_i$, where $\mu_i, \lambda_i$ are the meridian and longitudes for $J_i$ as a knot in $S^3$. (Note that $\mu_i = \alpha_i$.) We say that the slopes $\eta_i$ and $\alpha_i$ are dual to each other. We calculate these in the following order:

1. First, perform surgery on all components of $L$ to get $J$ in $S^3$, not just some of the components (which still produces $S^3$ if $L$ is realizing an $n$-adjacency).
2. Then, identify the slopes $\eta_i$ with $r_i = p_i / q_i$.

Note that the adjacency is integral if and only if all $p_i / q_i$ are integral. This is because $\Delta(\eta_i, \alpha_i) = \Delta(\mu_i, p_i \mu_i + q_i \lambda_i) = |q_i|$. In general, when discussing the dual link of a surgery to $S^3$, we will assume that it naturally inherits these rational surgery slopes in $S^3$ as above.

Now, the data $(S^3, J, r)$ in $S^3$ actually recovers $(Y, L, \alpha)$. Performing surgery on all components of $J$ gives $Y$ and by construction $L$ is the core of the surgery on $J$ while the meridian of each $J_i$ becomes the slope $\alpha_i$. But, we can also recover sublinks in the following way. If we look at a sublink $J'$ of $J$, without loss of generality, $J_1 \cup \ldots \cup J_k$, then surgery on $J'$ produces the same manifold as surgery in $Y$ on $L_{k+1} \cup \ldots \cup L_n$. And further, the core of surgery on $J'$, a $k$-component link, is exactly the image of $L_1 \cup \ldots \cup L_k$ in the surgery on $L_{k+1} \cup \ldots \cup L_n$.

Now we return to the case that a rational surgery on $L$ is realizing an $n$-adjacency from $Y$ to $S^3$. Build $(S^3, J, r)$ as discussed. The above paragraph can be reinterpreted as saying that doing the corresponding surgery on every proper sublink of $J$ gives $S^3$. For the benefit of the reader, we summarize this discussion with the following proposition.

**Proposition 2.2.** Let $\alpha$ be a multi-slope on an $n$-component link $L$ in $Y$. Then $(Y, L, \alpha)$ realizes an $n$-adjacency to $S^3$ if and only if there exists a multi-slope $\beta$ on a link $J$ in $S^3$ such that:

1. $S^3_\beta(J) = Y$;
2. $L$ is the core of the surgery on $J$;
3. each $\alpha_i$ is the dual slope to $\beta_i$;
4. surgery on every proper sublink of $J$ yields $S^3$.

Furthermore, the adjacency is integral if and only if $\beta$ is integral.
2.2. The linking of the dual curves. In light of the dual perspective from Proposition 2.2, we want to understand the effects of surgery on links in $S^3$ whose sublinks also surger to $S^3$. The next lemma allows us to constrain the linking numbers and surgery coefficients for the dual link in $S^3$ arising from an $n$-adjacency.

Lemma 2.3. Suppose that $(S^3, J, \alpha)$ realizes a 2-adjacency to $S^3$. Then either the linking number of $J$ is zero, or the linking number is $\pm 1$ and the surgery coefficients $\alpha_i$ are $\pm (1, 1/2)$ or $\pm (1/2, 1)$.

Proof. Since surgery on each individual component of $J = J_1 \cup J_2$ produces $S^3$, an integer homology sphere, the surgery coefficient for $J_i$ is of the form $1/q_i$. The linking matrix for the surgery presentation on $J$ then gives

$$1 = \left| \det \begin{pmatrix} 1 & q_2 \ell \\ q_1 \ell & 1 \end{pmatrix} \right| = |q_1 q_2 \ell^2 - 1|,$$

where $\ell$ is the linking number of $J_1$ and $J_2$ (say after choosing orientations of each component). Since $q_1, q_2 \neq 0$, we see that $|\ell| = 0$ or $1$. If $|\ell| = 1$, then we must have that $(q_1, q_2) = (\pm (1, 2))$ or $(\pm (2, 1))$, as desired. \hfill $\Box$

Proposition 2.4. Suppose that $J$ is an $n$-component link in $S^3$, with $n \geq 3$, and $\alpha$ is a multi-slope on $J$. If surgery on every proper sublink of $J$ produces $S^3$, then all pairwise linking numbers are 0 or $\pm 1$. If $J_1$ and $J_2$ are a pair of components with $|\ell k(J_1, J_2)| = 1$, then the slopes are $\pm (1, 1/2)$ or $\pm (1/2, 1)$. If $n = 3$ and $S^3_\alpha(J)$ is an integer homology sphere or $n \geq 4$, then each of $J_1$ and $J_2$ has linking number zero with all other components.

Proof. Since $n \geq 3$, every two-component sublink of $J$ with induced multi-slope from $\alpha$ provides a 2-adjacency from $S^3$ to itself. Therefore, by Lemma 2.3, the pairwise linking numbers are 0 or $\pm 1$ and for the 2-component sublinks with linking number having absolute value 1, the surgery coefficients are $\pm (1, 1/2)$ or $\pm (1/2, 1)$.

Now suppose that $S^3_\alpha(J)$ is an integer homology sphere. It remains to consider the pairwise linking of $J_1, J_2$ with the other components. Let $J_3$ be another component. Then, we know that the associated surgery on $J_1 \cup J_2 \cup J_3$ produces $S^3_\alpha(J)$ if $n = 3$ and $S^3$ if $n > 3$. Either way, the result is an integer homology sphere. Orient $J_1, J_2$ such that the pairwise linking is 1, fix an orientation on $J_3$ and let $\ell_1, \ell_2$ be the linking numbers of $J_3$ with $J_1, J_2$ respectively. Without loss of generality, the surgery coefficients on $J_1$ and $J_2$ are 1 and 1/2 respectively. (Otherwise, rearrange the order of the components and/or mirror $J$ and reverse the signs of $\alpha$.) Suppose for contradiction that $\ell_1, \ell_2$ are not both zero.

The first case is that $\ell_1 \neq 0$. In this case, by applying the first part of the proposition to the pair $(J_1, J_3)$, we see that $\ell_1 = 1$ and the surgery coefficient for $J_3$ must be 1/2. Applying the first part of the proposition to the pair $(J_2, J_3)$, we see that $J_2$ and $J_3$ have linking number zero, since the pair of surgery coefficients is not $\pm (1/2, 1)$ or $\pm (1, 1/2)$.
In this case, the linking matrix for the 3-component surgery description computes the order of $H_1$ of the surgery on $J_1 \cup J_2 \cup J_3$ to be:

$$1 = \det \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 3,$$

a contradiction.

The other case is that $\ell_1 = 0$, and so $\ell_2 = 1$. Now we see the surgery coefficients are 1 for $J_1$ and $J_3$, and $1/2$ for $J_2$. Then, we compute again

$$1 = \det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} = 3,$$

another contradiction. This completes the proof. \qed

3. Self-adjacencies from $S^3$

In this section, we constrain the self-adjacencies from $S^3$ to itself. As a warm-up, we begin with a special case.

**Proposition 3.1.** Suppose $(S^3, J, \alpha)$ realizes an $n$-adjacency to $S^3$ and the pairwise linking numbers of $J$ vanish. Then $J$ is the unlink and $\alpha_i = 1/k_i$ for $k_i \in \mathbb{Z}^*$ for all $i$.

**Proof.** First, it is clear that $\alpha_i = 1/k_i$ for each $i$ by a homological computation. We proceed by induction to show that $J$ is the unlink. The case of $n = 1$ is handled by the knot complement theorem [GL89]. Next we do the case of a 2-component link, $J_1 \cup J_2$. From the $n = 1$ case, each component is unknotted. Since $1/k_2$-surgery on $J_2$ is $S^3$, the image of $J_1$ in $1/k_2$-surgery on $J_2$ has a surgery to $S^3$. Hence, the image of $J_1$ is also unknotted. In particular, the component $J_2$ in the complement of $J_1$ in $S^3$ is a knot in a solid torus which has a non-trivial solid torus surgery. By [Gab89], $J_1$ is contained in a ball or is a braid in the solid torus, so has non-zero winding number. However, because $\ell k(J_1, J_2) = 0$, it must be the case that $J_1$ is contained in a ball in the complement of $J_2$ in $S^3$, meaning $J_1$ and $J_2$ are unlinked. This completes the proof for $n = 2$ components.

For the inductive step, our hypothesis is that $J$ is a Brunnian link and then we will deduce that $J$ is in fact an unlink. This can be found in [GLLM22 Proposition 4.1], but for self-containedness, we present an elementary proof that does not rely on Heegaard Floer homology. Suppose the result is true for $(n-1)$-component links and that $(S^3, J, \alpha)$ realizes an $n$-adjacency to $S^3$. Then $J$ is Brunnian, $J_1 \cup \ldots \cup J_{n-1}$ is an unlink, and so $(1/k_1, \ldots, 1/k_{n-1})$-surgery on $J_1 \cup \ldots \cup J_{n-1}$ gives $S^3$. Thus the image of $J_n$ must be unknotted after this surgery. Hence, we see that $(1/k_1, \ldots, 1/k_{n-1}, 1/m)$-surgery on $J$ gives $S^3$ for arbitrary $m$. For the sake of concreteness, fix $m = 5$.

We claim that the image of $J_1 \cup \ldots \cup J_{n-1}$ after performing $1/5$-surgery on $J_n$, denoted $K_1 \cup \ldots \cup K_{n-1}$, yields an $(n-1)$-component Brunnian link. To see that the $(n-1)$-component image link is Brunnian, note that all $(n-1)$-component sublinks of $J$ are unlinks by our inductive hypothesis. In particular, $J_1 \cup \ldots \cup J_{n-2} \cup J_n$ is an unlink,
hence 1/5-surgery along \( J_n \) shows that \( K_1 \cup \ldots \cup K_{n-2} \) is an unlink. A similar argument applies to the other \( n-2 \)-component sublinks of \( K_1 \cup \cdots \cup K_{n-1} \).

Thus, the link \( K_1 \cup \cdots \cup K_{n-1} \) is a Brunnian link with a surgery to \( S^3 \), and so is an unlink by induction. In other words, 1/5-surgery on \( J_n \) in the exterior of \( J_1 \cup \ldots \cup J_{n-1} \) produces a reducible 3-manifold (the exterior of an \( (n-1) \)-component unlink). Yet, the distance between the trivial slope \( \infty \) and 1/5 is 5. A theorem of Gordon and Litherland [GLS4] Theorem 1.1 implies that for any pair of reducible Dehn fillings on an irreducible manifold, the slopes have distance at most four. This is a contradiction. This implies that the exterior of \( J \) must be reducible. Hence, \( J \) is split. However, a split Brunnian link is an unlink. \( \square \)

We now build on the previous proposition to complete our characterization of the self-adjacencies of \( S^3 \) promised in the introduction.

**Proposition 3.2.** The triple \( (S^3, J, \alpha) \) realizes an \( n \)-adjacency to \( S^3 \) if and only if \( J \) itself is a split union of Hopf links and unknots, all slopes \( \alpha_i = 1/k_i \), where \( k_i \in \mathbb{Z}^* \), and the surgery slopes of Hopf components are either \( \pm(1, 1/2) \) or \( \pm(1/2, 1) \).

**Proof.** As in the proof of Proposition 3.1 the knot complement theorem implies that the components are unknotted and of course the surgery coefficients are of the form \( 1/k_i \).

We begin with the case of \( n = 2 \). Consider \( J_1 \cup J_2 \). The case that \( \ell k(J_1, J_2) = 0 \) follows from Proposition 3.1. By Lemma 2.3 we assume \( \ell k(J_1, J_2) = 1 \) and the surgery coefficients are \( \pm(1, 1/2) \). Note that \( J_1 \) is unknotted in 1/2-surgery on \( J_2 \), so \( J_2 \) can again be viewed as a knot in the solid torus with a solid torus surgery. Therefore, by [Gab89], \( J_2 \) is a braid in the complement of \( J_1 \) and the winding number is the linking number of \( J_1 \) and \( J_2 \). The only winding number 1 braid in the solid torus is the core. We see that \( J_1 \cup J_2 \) is a Hopf link.

Next, we handle the case of \( n = 3 \). If the pairwise linking numbers are zero, we appeal to Proposition 3.1. So, assume some pair of components \( J_1, J_2 \) have nonzero linking number. By Proposition 2.4, \( J_1, J_2 \) have surgery coefficients \( \pm(1, 1/2) \) and linking number 1. Further, by Proposition 2.4 we have that \( J_3 \) is algebraically split from \( J_1 \) and \( J_2 \). By the \( n = 2 \) case of the proof, \( J_1 \cup J_2 \) must form a Hopf link. We also have that \( J_3 \) is an unknot that is geometrically split from \( J_1 \) and \( J_2 \) individually, but possibly not split from the link \( J_1 \cup J_2 \).

It remains to show that \( J_3 \) is in fact split from \( J_1 \cup J_2 \). Since \( J_1 \cup J_3 \) is an unlink, trivial surgery on \( J_2 \) produces the 2-component unlink \( J_1 \cup J_3 \). Also, the image of \( J_1 \cup J_3 \) under 1/2-surgery on \( J_2 \) is a 2-component unlink because this image is a 2-component link, all of whose induced surgeries give \( S^3 \). We now appeal to [CGLS87], Corollary 2.4.7. This states that if an irreducible three-manifold admits two reducible Dehn fillings along slopes of distance at least two on a torus boundary component, one of the filled manifolds contains a lens space summand. However, a link complement in \( S^3 \) cannot contain a lens space summand. As the surgered manifolds are link complements in \( S^3 \), the exterior of \( J \) is reducible, and so \( J \) is a split link.
Now, we complete the induction using a similar strategy. Suppose that \( J \) has \( n \) components. By assumption, all proper sublinks are split unions of unlinks and Hopf links. If \( J \) has pairwise linking numbers all zero, we can again apply Proposition 3.1. Therefore, up to reordering of the components, we have at least one two-component sublink \( J_1 \cup J_2 \) which is a Hopf link. Up to mirroring and reordering \( J_1 \) and \( J_2 \), the surgery coefficient on \( J_1 \) is \( 1/2 \). Trivial surgery on \( J_2 \) produces a split link as does \( 1/2 \)-surgery, since the resulting link is an \((n-1)\)-component link satisfying the same hypotheses of the theorem. Therefore, by appealing again to [CGLS87, Corollary 2.4.7], we get that the complement of \( J \) is reducible, so \( J \) is split. Since all \((n-1)\)-component sublinks are split unions of Hopf links and unknots, and because \( J \) itself is split, we now have that \( J \) is a split union of Hopf links and unknots.

\[ \square \]

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**References**


