On the Accuracy of Input-Output Uncertainty Modeling with Interval Type-2 Fuzzy Logic Systems

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Abstract— Type-2 Fuzzy Logic Systems (T2 FLSs) have been commonly attributed with the capability to model various sources of data uncertainties. The input uncertainties of an FLS were modeled using T2 Fuzzy Sets (FSs) and the type-reduced centroid of the output FS was interpreted as a measure of uncertainty associated with the terminal real-valued output. However, the accuracy of this input-output uncertainty modeling has been rarely studied. It is well established that T2 FSs can be understood as a composition of a large number of embedded T1 FSs and thus model the uncertainty of selecting a specific T1 FSs. However, whether the same can be achieved with T2 FLSs can be considered an open question. This paper contributes by presenting a study of the input-output uncertainty modeling capability of Interval T2 (IT2) FLSs. First, the Monte Carlo simulation technique is used to simulate linguistic uncertainties and to compute the aggregated output result. This simulation is then compared to the output bounds provided by the interval centroid computed with IT2 FLS. It is demonstrated that the interval output of the IT2 FLS overestimates the output uncertainty range when compared to the results of the Monte Carlo simulation. To further understand this problem the concept of Equivalent Type-1 FSs is used. Finally, a detailed example is presented to demonstrate why the IT2 fuzzy inference process overestimates the output uncertainty.

***Index Terms*— Interval Type-2 Fuzzy Sets, Fuzzy Logic Systems, Centroid, Type-Reduction, Uncertainty Modeling**

I. Introduction

TYPE-2 Fuzzy Logic Systems (T2 FLSs) have been studied by many researchers in recent years [1]-[4]. T2 FLSs are based on the concepts of T2 Fuzzy Sets (FSs) originally proposed by Lofti Zadeh [5], The major difference between T1 and T2 FLSs is in the model of individual FSs, where T2 FSs use membership degrees that are themselves FSs. It has been shown that T2 FLSs can improve the performance of T1 FLSs, especially when applied to problems with various data uncertainties [6]-[9]. The various sources of uncertainty are commonly identified as follows: i) uncertainty in the linguistic knowledge used to construct the FLS, ii) uncertainty about the correct outputs of the system, iii) uncertainty associated with noisy inputs, and iv) uncertainty about the data that were used to tune the parameters of the control system. This paper focuses on modeling of the linguistic uncertainty.

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The most widely used kind of T2 FLSs is the Interval T2 (IT2) FLS, which uses IT2 FSs with constrained interval membership degrees [10]. Many researchers argue in favor of IT2 FLSs because of their potential to model and minimize the effects of uncertainties, while providing computationally efficient framework when compared to General T2 FLS [3], [6], [10]. Typically, the performance of the IT2 FLSs is compared to their T1 counterparts demonstrating improvements when noise and other types of uncertainties are introduced into the system. The improved performance can be attributed to the Footprint of Uncertainty (FOU) of the IT2 FSs, which allows for improved modeling of the input uncertainties.

The results of the IT2 fuzzy inference process is an output IT2 FS. This IT2 FS must be first type-reduced into its interval centroid, which can then be defuzzified into the terminal real-valued output [11]. Frequently, the geometric properties of the output interval centroid are associated with the uncertainty about the system's real-valued output [12]-[24]. For instance, Wu and Mendel state in [12] that: *"...the length of the type-reduced set can therefore be used to measure the extent of the output's uncertainty.*" In addition, the concept of embedded T1 FLS is defined in [12] and a T2 FLS is interpreted as a collection of its embedded T1 FLSs. Ren et al. used the interval output of IT2 FLS to model and predict the variations of micro milling cutting forces [13]-[15]. The variations of the width of the interval centroid as a result of various FOUs for the input IT2 FSs was studied by Ozen, Garibaldi and others [16]-[18]. In [19], the interval centroid was used to establish an uncertainty band around the output of perceptual computer and to guide the process of choosing optimal location for international logistics centers. Other researchers used the interval centroid to establish an uncertainty measure on the output of a system in applications such as stock price analysis, short-term traffic forecasting or modeling of photovoltaic arrays [20]-[23]. In these examples, the IT2 FLS is assumed to perform an input-output uncertainty mapping. The uncertainty modeled by the IT2 FSs is thus assumed to be reflected in a specific uncertainty measure associated with the interval output [24].

However, the accuracy of this input-output uncertainty mapping has been rarely studied. It is well established that IT2 Fuzzy Sets can be understand as a composition of a large number of embedded T1 FSs. IT2 FSs thus model the uncertainty associated with selecting a specific T1 FS from within the FOU [24]. It has been previously shown that the interval centroid of an IT2 FS accurately captures the range of

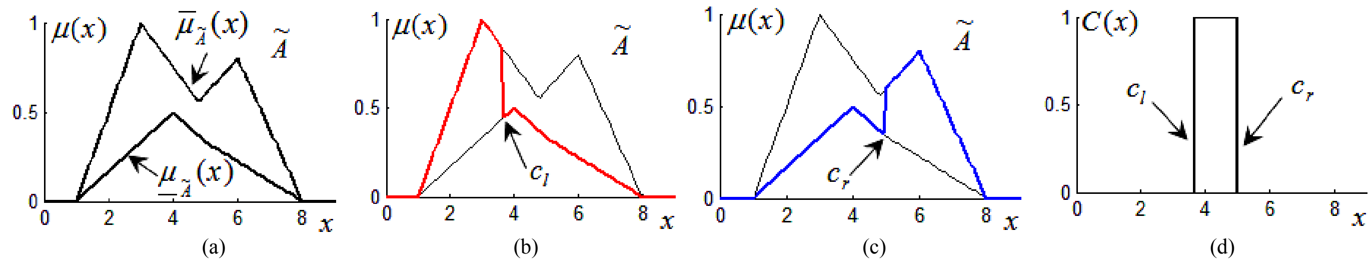


Fig. 1 FOU of an IT2 FS $\tilde{A}$ (a), its left (b) and right (c) boundary embedded T1 FSs and the interval centroid $C=[c_l, c_r]$ (d).

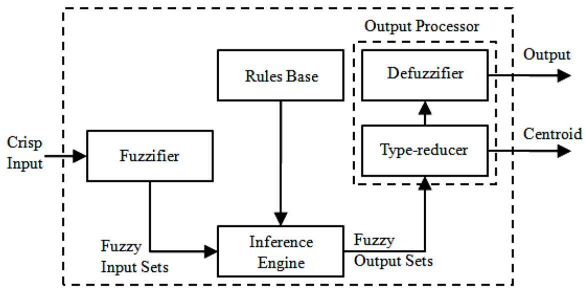


Fig. 2 IT2 FLS [4].

possible defuzzified values given the uncertain selection of specific T1 membership function. This paper investigates whether the same can be claimed about IT2 Fuzzy Logic Systems. The main contribution lies in the analysis of the input-output uncertainty modeling capability of IT2 FLSs. First, a Monte-Carlo (MC) technique is used to calculate an interval output of a perturbed T1 FLSs simulating the effects of linguistic uncertainties. This result is then compared to the output interval centroid of an IT2 FLS, which uses IT2 FSs that model identical linguistic uncertainty. The experiment identifies inconsistencies between the MC uncertainty simulation and the IT2 FLS interval output. The IT2 FLS is found to overestimate the output uncertainty in specific scenarios.

To further understand this problem the concept of Equivalent Type-1 FS (ET1 FS) is utilized [25]. The ET1 FS method searches for a T1 membership grade for a selected IT2 FS, which would ensure that the output value of the original IT2 FLS and the newly created T1 FLS remain the same. It is shown that the IT2 FLS overestimates the output uncertainty because the ET1 FSs are often located outside the FOU of the original IT2 FSs. Finally, a detailed example is presented to identify some of the causes for the overestimation of the output IT2 FLS uncertainty.

The rest of the paper is organized as follows. Section II briefly reviews the fundamentals of IT2 FLSs and defines the experimental IT2 FLS used. Section III presents the comparison of input-output linguistic uncertainty modeling using MC simulation and IT2 FLS. The concept of ET1 FS is explained and used for further analysis in Section IV. Section V presents a detailed example and the paper is concluded in Section VI.

II. Interval Type-2 Fuzzy Logic Systems

This section provides brief overview of IT2 FLS. Next, an exemplary IT2 FLS used in the rest of this paper is described.

A. Interval Type-2 Fuzzy Logic Systems

An IT2 FS $\tilde{A}$ can be expressed as [3]:

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u) \qquad J_x \subseteq [0,1] \tag{1}$$

Here, x and u are the primary and secondary variables, X is the domain of variable x and J_x is the primary membership of x. In the special case of IT2 FSs, all secondary grades of fuzzy set $\tilde{A}$ are equal to 1. The domain of the primary memberships J_x defines the FOU of $\tilde{A}$. The FOU of an IT2 FS $\tilde{A}$ can be bounded by its upper and lower membership functions (see Fig. 1(a)):

$$FOU(\tilde{A}) = \bigcup_{\forall x\in X} (\underline{\mu}_{\tilde{A}}(x),\ \overline{\mu}_{\tilde{A}}(x)) \tag{2}$$

The IT2 FS $\tilde{A}$ can be type-reduced into its centroid $C_{\tilde{A}}$, which itself is an interval T1 FSs bounded by its left and right boundaries $C_{\tilde{A}} = [c_l, c_r]$. It has been shown that these boundaries can be obtained by solving the following optimization problems [11]:

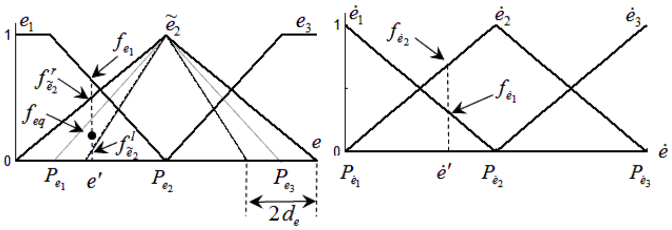
$$c_l = \min_{\forall \theta_i \in [\underline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{A}}(x_i)]} \frac{\sum_{i=1}^{N} x_i\,\theta_i}{\sum_{i=1}^{N} \theta_i} \tag{3}$$

$$c_r = \max_{\forall \theta_i \in [\underline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{A}}(x_i)]} \frac{\sum_{i=1}^{N} x_i\,\theta_i}{\sum_{i=1}^{N} \theta_i} \tag{4}$$

Here, symbol θ denotes an auxiliary variable that is used to select a specific membership grade from within the primary membership. Hence, the interval centroid captures the uncertainty associated with the terminal output value for a random selection of an embedded T1 FS from within the FOU [24]. The boundary embedded fuzzy sets c_l and c_r and the final interval centroid are depicted in Fig. 1.

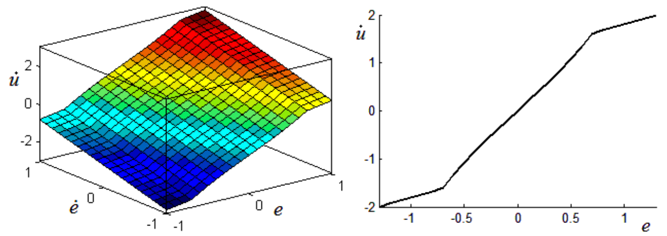
The IT2 FLS is composed of four major parts: input fuzzifier, fuzzy inference engine, fuzzy rule base and output processor, as depicted in Fig. 2 [4]. The IT2 FLS works similarly when compared to T1 FLS, with the exception that at least one fuzzy set must be an IT2 FS. Fuzzy interval meet and join operations are then used to calculate the firing strengths and to aggregate the outputs of individual fuzzy rules encoded in an implicative form as follows:

IF x_1 is $\tilde{A}_1^k$ **AND** … **AND** x_n is $\tilde{A}_n^k$ **THEN** y_k is $\tilde{B}^k$ (5)



(a) (b)

Fig. 3 Inputs of the IT2 FLS e (a) and $\dot{e}$ (b).



(a) (b)

Fig. 4 Output surface of the IT2 FLS (a) and a slice of the output surface for $\dot{e} = 0.0$ (b).

TABLE I
FUZZY RULE TABLE

$e / \dot{e}$	$\dot{e}_1$	$\dot{e}_2$	$\dot{e}_3$
e_1	$\dot{u}_{11}$	$\dot{u}_{12}$	$\dot{u}_{13}$
$\tilde{e}_2$	$\dot{u}_{21}$	$\dot{u}_{22}$	$\dot{u}_{23}$
e_3	$\dot{u}_{31}$	$\dot{u}_{32}$	$\dot{u}_{33}$

Here, symbol $\widetilde{A}_j^k$ and $\widetilde{B}^k$ denote the j^{th} input IT2 FS and the output IT2 FS, n is the dimensionality of the input vector $\bar{x}$, and y_k is the associated output variable.

The output processor first performs type-reduction operation, which reduces the output IT2 FSs into its T1 interval centroid [11]. The centroid can then be defuzzified to produce the terminal real-valued output.

B. IT2 Fuzzy PI controller

In the rest of this paper an exemplary IT2 FLS is used. The structure of this IT2 FLS is consistent with the IT2 fuzzy Proportional-Integral controller presented in [25]. The implemented PI control law can be stated as $u(t) = K_P e(t) + K_P \int_0^t e(\tau)\, d\tau$. The IT2 FLS has two inputs, the error signal e and the rate of error change $\dot{e}$. The output is the control signal change $\dot{u}$. Each input domain is characterized using three input FS parametrized by points $\{P_{e_1}, P_{e_2}, P_{e_3}\}$ and $\{P_{\dot{e}_1}, P_{\dot{e}_2}, P_{\dot{e}_3}\}$ as depicted in Fig. 3. In this specific implementation all input fuzzy sets are of type-1 except for IT2 FS $\tilde{e}_2$ as shown in Fig. 3(a). Parameter d_e controls the width of the FOU of IT2 FS $\tilde{e}_2$. This specific design was selected for its suitability for the demonstration of input-output uncertainty mapping using the equivalent fuzzy sets as described later in Section IV.

The fuzzy rule base contains nine fuzzy rules, one for each combination of the input FS. The consequents are singleton values computed as follows:

$$\dot{u}_{ij} = K_I P_{e_i} + K_P P_{\dot{e}_j} \quad i, j = 1,2,3 \tag{6}$$

Here, K_I and K_P are the gain coefficients of the fuzzy PI controller. The fuzzy rule base is summarized in Table I.

The inference process uses product t-norm to calculate the rule firing strength. The iterative KM algorithm is used to compute the interval centroid of the output IT2 FS [26]. An example of a control surface for the described IT2 FLS with parameters $K_I = 0.5$, $K_P = 1.0$ and $d_e = 0.5$ and its slice at input value $\dot{e} = 0.0$ is shown in Fig. 4.

III. IT2 FLS Uncertainty Modeling Analysis using Monte Carlo Simulation

This section first outlines two approaches for input-output linguistic uncertainty modeling of FLS using Monte Carlo (MC) simulation and using IT2 fuzzy logic. Next, an experimental comparison of both approaches is presented and analyzed.

A. Uncertainty modeling using MC simulation and T1 FLS

It is well established that T2 FSs can be understood as a composition of a large number of embedded T1 FSs [24]. T2 FSs model the uncertainty associated with the selection of a specific embedded T1 FS. In a similar manner, it is commonly assumed that IT2 FLS act as a composition of a large number of T1 FLS, each with different T1 FSs selected from within the FOUs of the original IT2 FSs [12], [27]. The interval output computed as the interval centroid is then interpreted as the uncertainty associated with the output value due to the uncertain choice of specific T1 fuzzy membership functions.

Previously, the sensitivity and robustness due to input uncertainties of T1 FLSs was analyzed using MC simulations [28], [29]. In the presented paper, the MC simulation is utilized to model the effects of linguistic uncertainty and to compute the range of output values given an uncertain choice of specific input T1 FSs. At each iteration, a random embedded T1 FS is selected from within the FOU of the input IT2 FSs and the output value for the obtained T1 FLS is calculated. The results from a large number of iterations are aggregated and the minimum and the maximum output values are computed resulting in an output uncertainty interval. This process is illustrated in Fig. 5(a).

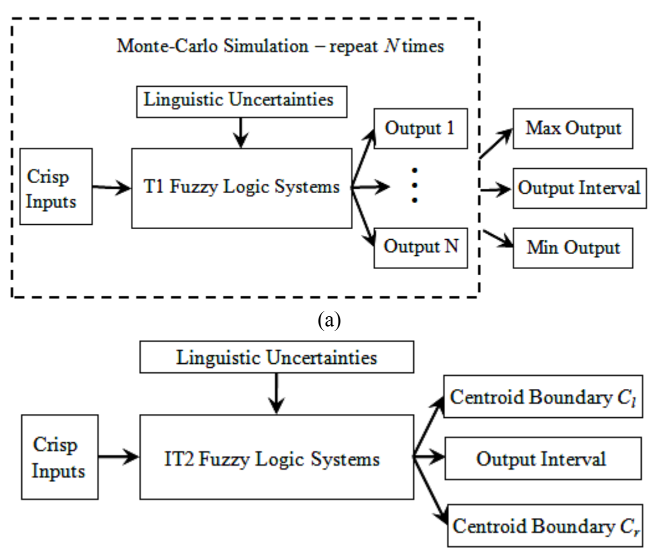


Fig. 5 Modeling linguistic uncertainties with Monte-Carlo simulation and T1 FLS (a) and with IT2 FLS (b).

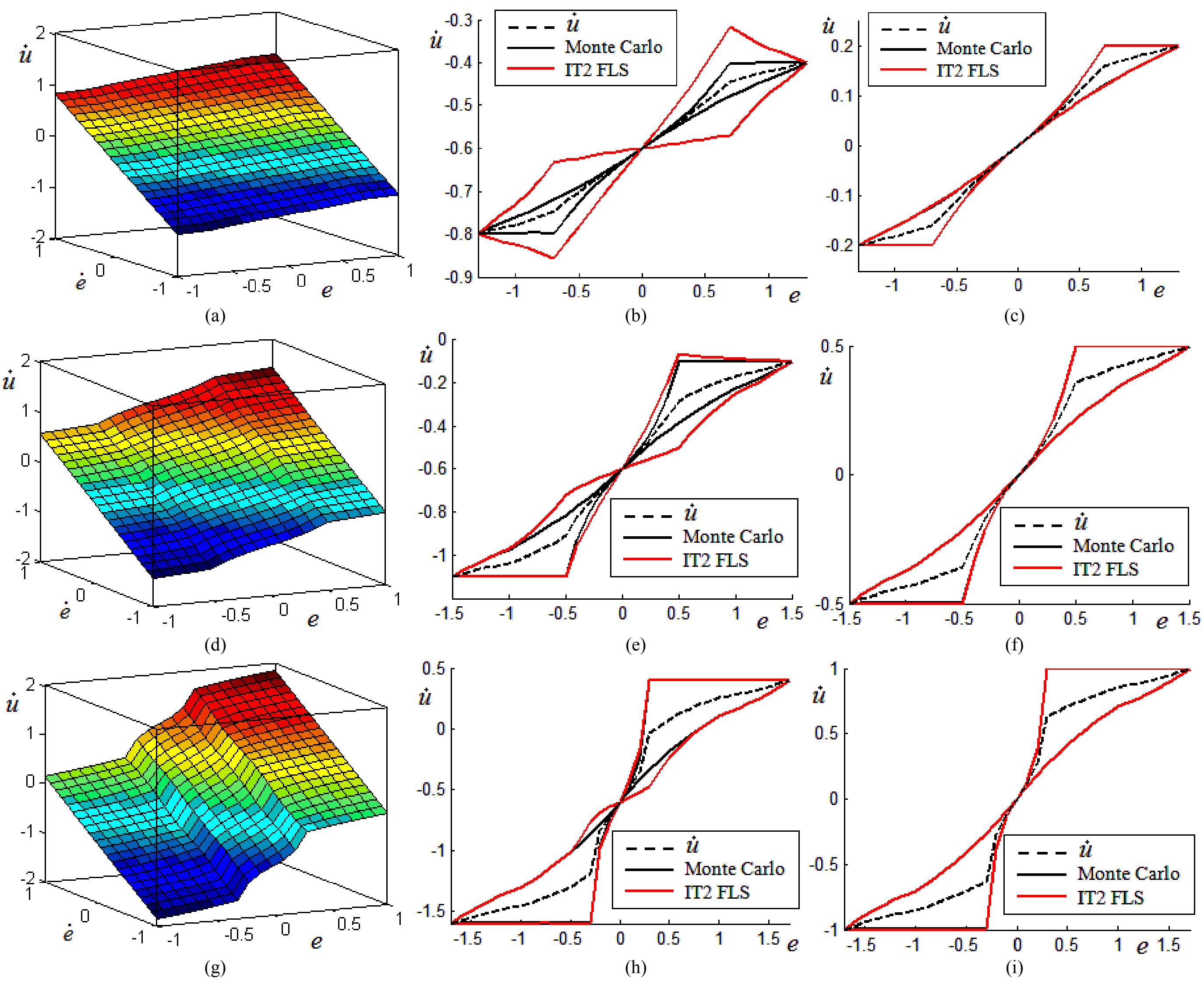


Fig. 6 Output surface and the output uncertainty for output surface slices at $\dot{e} = -0.6$ and $\dot{e} = 0.0$ for IT2 FLS with $K_I = 0.2$ and $d_e = 0.3$ (a)-(c), $K_I = 0.5$ and $d_e = 0.5$ (d)-(f), and for $K_I = 1.0$ and $d_e = 0.7$ (g)-(i).

B. Uncertainty modeling using IT2 FLS

The IT2 FLS models the uncertainty associated with selecting a specific membership grade using the FOU of the input IT2 FSs. The interval arithmetic used by the fuzzy inference process computes the output IT2 FS. During the output processing stage, the output IT2 FS is type-reduced into an interval centroid C described using its left and right boundaries as $C = [c_l, c_r]$. As discussed previously, the centroid boundaries are commonly assumed to express the uncertainty associated with the terminal output value. This input-output uncertainty modeling is depicted in Fig. 5(b).

C. Comparison

The implemented IT2 fuzzy PI controller described in Section II.B was used for the experimental comparison of the two distinct approaches to input-output uncertainty modeling of FLS. The MC simulation was run with the number of samples N=1000. For each iteration, a random embedded T1 FS was uniformly selected from within the FOU of the IT2 FS $\tilde{e}_2$, as shown in Fig. 3(a). Different configurations of the FLS were used in the study with the following parameters:

$$P_{e_1} = -1,\ P_{e_2} = 0,\ P_{e_3} = 1 \tag{7}$$

$$P_{\dot{e}_1} = -1,\ P_{\dot{e}_2} = 0,\ P_{\dot{e}_3} = 1 \tag{8}$$

$$K_P = 1,\ \ K_I = \{0.2, 0.5, 1.0\},\ \ d_e = \{0.3, 0.5, 0.7\} \tag{9}$$

First, the IT2 FLS with $K_I = 0.2$ and $d_e = 0.3$ was tested. Fig. 6(a) shows the output surface. Fig. 6(b) and Fig. 6(c) depict the comparison of the interval output produced by IT2 FLS and the interval output produced by MC simulation for two slices of the output surface at $\dot{e} = -0.6$ and $\dot{e} = 0.0$. It can be observed that for the surface slice at $\dot{e} = -0.6$ there is a significant inconsistency between the two methods for computing the output intervals. Namely, the IT2 FLS overestimates the range of output uncertainty. On the other hand, for the surface slice at $\dot{e} = 0.0$ both uncertainty models

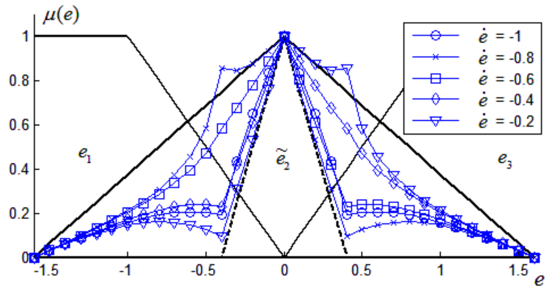


Fig. 7 ET1 FS for $\tilde{e}_2$ for different slices of the output surface.

correspond very well and the IT2 FLS correctly matches the MC simulation.

Next, the IT2 FLS with $K_I = 0.5$ and $d_e = 0.5$ was tested. Fig. 6(d) shows the output surface and the control surface slices at values $\dot{e} = -0.6$ and $\dot{e} = 0.0$ are depicted in Fig. 6(e) and Fig. 6(f). Again, it can be observed that the IT2 FLS overestimates the range of output uncertainty for $\dot{e} = -0.6$.

Finally, identical observations can be made by analyzing the output surface and the surface slices for IT2 FLS with $K_I = 1.0$ and $d_e = 0.7$ depicted in Fig. 6(g)-(i).

IV. IT2 FLS Uncertainty Modeling Analysis Using Equivalent Type-1 Fuzzy Sets

This section first reviews the concept of Equivalent Type-1 (ET1) FSs proposed by Wu and Tan [25]. Next, the ET1 FSs method is used to further analyze the input-output uncertainty modeling of IT2 FLSs.

A. Equivalent Type-1 Fuzzy Sets

The concept of ET1 FSs was originally proposed by Wu and Tan and used to perform the IT2 FLS modeling capability analysis [25]. The key idea of ET1 FSs is that the IT2 FSs of an IT2 FLS can be reduced to a group of T1 FSs without affecting the output of the IT2 FLS. In other words, for a specific input values, the ET1 FSs method seeks to find such a T1 fuzzy membership grades, which when substituted for the IT2 fuzzy membership result in an unchanged output value. The existence and uniqueness of ET1 FSs has been proven in [25].

Consider, the IT2 fuzzy PI controller introduced in Section II.B. Assume that the firing strengths for the T1 FSs e_1, e_2, $\dot{e}_1$ $\dot{e}_2$ and $\dot{e}_3$ are denoted as f_{e_1}, f_{e_3}, $f_{\dot{e}_1}$, $f_{\dot{e}_2}$ and $f_{\dot{e}_3}$ and the interval firing strength of IT2 FS $\tilde{e}_2$ is denoted as $f_{\tilde{e}_2}$ as labeled in Fig. 3. Then the firing strength R^{ij} of individual rules obtained using the product t-norm can be expressed as:

$$R^{ij}: \begin{cases} f_{e_i} \times f_{\dot{e}_j} & i = 1,3,\ j = 1,2,3 \\ f_{\tilde{e}_i} * f_{\dot{e}_j} = [f_{e_l} \times f_{\dot{e}_j}, f_{e_r} \times f_{\dot{e}_j}] & i = 2,\ j = 1,2,3 \end{cases} \quad (10)$$

The IT2 fuzzy inference can be used to compute the output value u. Consequently, the ET1 FS membership grade f_{eq} is to be found so that an identical output value is produced when f_{eq} is used in place of the interval firing strength $f_{\tilde{e}_2}$. The equivalent output of the resulting T1 FLS can be expressed as in (11). Finally, the membership degree of the ET1 FS can be

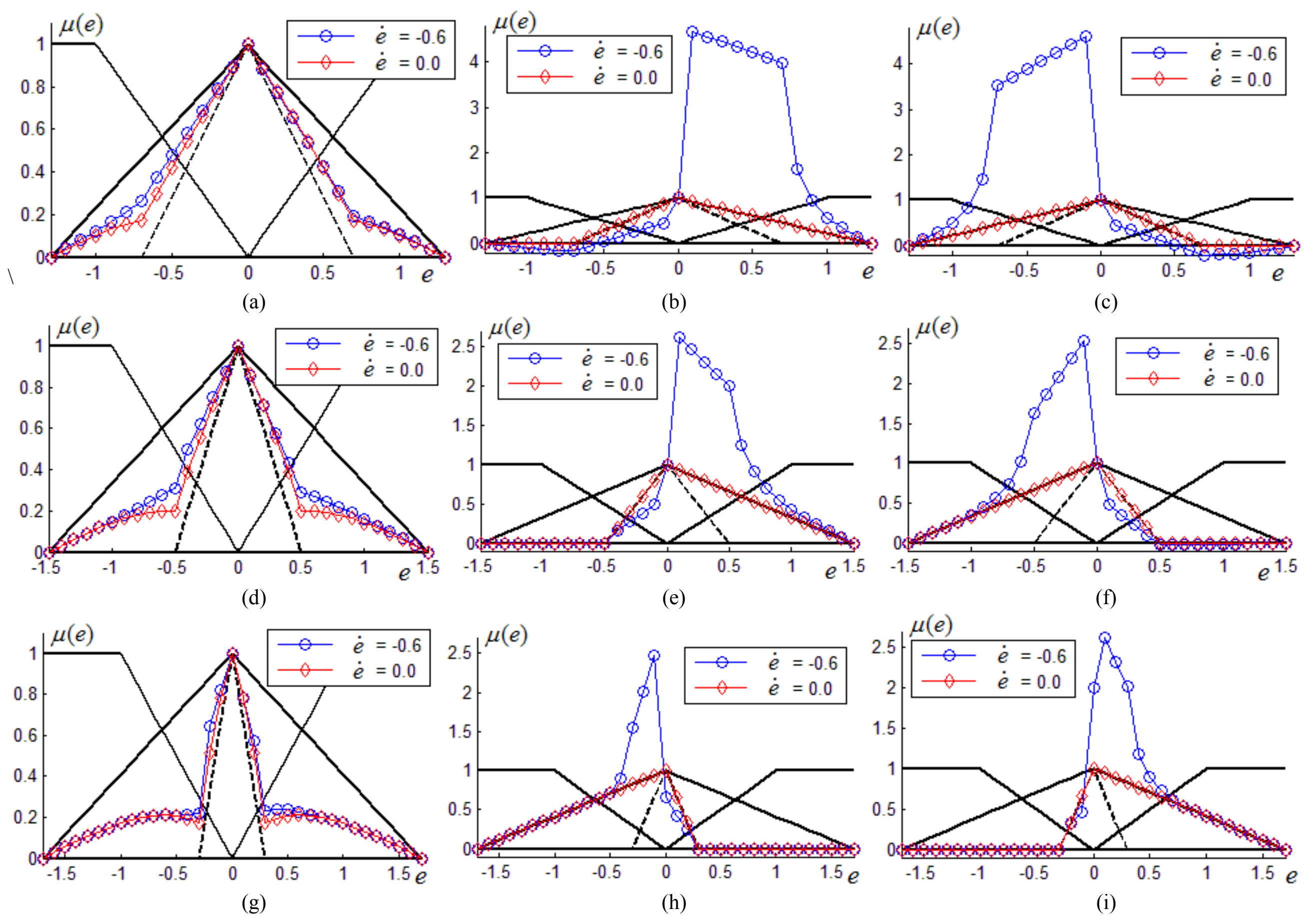


Fig. 8 ET1 FS for the output value u and the centroid boundary values c_l and c_r for IT2 FLS with $K_I = 0.2$ and $d_e = 0.3$ (a)-(c), $K_I = 0.5$ and $d_e = 0.5$ (d)-(f), and for $K_I = 1.0$ and $d_e = 0.7$ (g)-(i).

$$\dot{u} = \frac{f_{e1}f_{\dot{e}1}\dot{u}_{11} + f_{e1}f_{\dot{e}2}\dot{u}_{12} + f_{e1}f_{\dot{e}3}\dot{u}_{13} + f_{eq}f_{\dot{e}1}\dot{u}_{21} + f_{eq}f_{\dot{e}2}\dot{u}_{22} + f_{eq}f_{\dot{e}3}\dot{u}_{23} + f_{e3}f_{\dot{e}1}\dot{u}_{31} + f_{e3}f_{\dot{e}2}\dot{u}_{32} + f_{e3}f_{\dot{e}3}\dot{u}_{33}}{f_{e1}f_{\dot{e}1} + f_{e1}f_{\dot{e}2} + f_{e1}f_{\dot{e}3} + f_{eq}f_{\dot{e}1} + f_{eq}f_{\dot{e}2} + f_{eq}f_{\dot{e}3} + f_{e3}f_{\dot{e}1} + f_{e3}f_{\dot{e}2} + f_{e3}f_{\dot{e}3}} \quad (11)$$

$$f_{eq} = \frac{\dot{u}(f_{e1} + f_{e3})(f_{\dot{e}1} + f_{\dot{e}2} + f_{\dot{e}3}) - f_{e1}(f_{\dot{e}1}\dot{u}_{11} + f_{\dot{e}2}\dot{u}_{12} + f_{\dot{e}3}\dot{u}_{13}) - f_{e3}(f_{\dot{e}1}\dot{u}_{31} + f_{\dot{e}2}\dot{u}_{32} + f_{\dot{e}3}\dot{u}_{33})}{f_{\dot{e}1}\dot{u}_{21} + f_{\dot{e}2}\dot{u}_{22} + f_{\dot{e}3}\dot{u}_{23} - \dot{u}(f_{\dot{e}1} + f_{\dot{e}2} + f_{\dot{e}3})} \quad (12)$$

$$f_{eq}^{l} = \frac{c_l(f_{e1} + f_{e3})(f_{\dot{e}1} + f_{\dot{e}2} + f_{\dot{e}3}) - f_{e1}(f_{\dot{e}1}\dot{u}_{11} + f_{\dot{e}2}\dot{u}_{12} + f_{\dot{e}3}\dot{u}_{13}) - f_{e3}(f_{\dot{e}1}\dot{u}_{31} + f_{\dot{e}2}\dot{u}_{32} + f_{\dot{e}3}\dot{u}_{33})}{f_{\dot{e}1}\dot{u}_{21} + f_{\dot{e}2}\dot{u}_{22} + f_{\dot{e}3}\dot{u}_{23} - c_l(f_{\dot{e}1} + f_{\dot{e}2} + f_{\dot{e}3})} \quad (13)$$

$$f_{eq}^{r} = \frac{c_r(f_{e1} + f_{e3})(f_{\dot{e}1} + f_{\dot{e}2} + f_{\dot{e}3}) - f_{e1}(f_{\dot{e}1}\dot{u}_{11} + f_{\dot{e}2}\dot{u}_{12} + f_{\dot{e}3}\dot{u}_{13}) - f_{e3}(f_{\dot{e}1}\dot{u}_{31} + f_{\dot{e}2}\dot{u}_{32} + f_{\dot{e}3}\dot{u}_{33})}{f_{\dot{e}1}\dot{u}_{21} + f_{\dot{e}2}\dot{u}_{22} + f_{\dot{e}3}\dot{u}_{23} - c_r(f_{\dot{e}1} + f_{\dot{e}2} + f_{\dot{e}3})} \quad (14)$$

computed as in (12) upon rearranging (11).

As an example, the ET1 FSs for different slices of the output control surface are depicted in Fig. 7. It can be observed that for input values $\dot{e} = -0.8$ and $\dot{e} = -0.2$ some membership grades lie outside the FOU of the IT2 FS $\tilde{e}_2$.

B. Uncertainty Modeling Analysis using ET1 FS

The concept of ET1 FS was originally developed to investigate the differences in input-output modeling between T1 and IT2 FLS. In this paper, the ET1 FS method is applied to the analysis of the input-output uncertainty modeling performed by IT2 FLS. As demonstrated in Section III.C the interval output of the IT2 FLS appears to overestimate the output uncertainty obtained with the MC simulation. The ET1 FS method is used here to provide further evidence of these phenomena.

To investigate the input-output uncertainty mapping, it is of interest to map the interval centroid back to an equivalent T1 input FSs. Assuming that the IT2 FLS accurately models the uncertainties associated with selecting specific T1 input FSs from within the FOUs, it can be expected that the bounds of the interval centroid should map back into a pair of embedded T1 FSs located within the original FOUs. Hence, the ET1 fuzzy membership grades f_{eq}^{l} and f_{eq}^{r} are computed for the left and right interval centroid bounds c_l and c_r in (13) and (14).

Fig. 8 depicts the ET1 FSs for the output values u and the interval centroids bounds c_l and c_r for the test cases considered in Fig. 6. Two important observations can be made. First, in many cases the calculated ET1 FSs f_{eq}^{l} and f_{eq}^{r} lie outside the FOUs of the original IT2 FSs. This observation verifies that the interval centroid overestimates the output uncertainty, because it represents uncertainty associated with wider FOU than the one actually used. Second, the ET1 FS membership grades in many cases are greater than 1, which contradicts the basic definition of fuzzy sets. This second observation suggests that the boundaries of the interval centroid are not computable by any possible instantiation of the T1 FLSs. This last observation is consistent with a recent analysis presented in [30].

V. Understanding the IT2 FLS Input-Output Uncertainty Modeling

This section contains a detailed example that demonstrates the output uncertainty overestimation of IT2 FLS. Consider a controller with parameters as in (7) and (8) and $K_P = 1,\ K_I = 0.2,\ d_e = 0.3$ with control surface depicted in Fig. 6(a). For an input value pair $e = 0.5$ and $\dot{e} = -0.5$ the non-zero membership degrees will be as follows:

$$f_{\tilde{e}2} = [0.2857, 0.6154], f_{e3} = 0.5, f_{\dot{e}1} = 0.5, f_{\dot{e}2} = 0.5 \quad (15)$$

The four fired rules with their firing strengths obtained using the product t-norm can be summarized as:

Rule	Firing Strength	Consequent
R^{21}	$[0.1429, 0.3077]$	-1.0
R^{22}	$[0.1429, 0.3077]$	0.0
R^{31}	0.25	-0.8
R^{32}	0.25	0.2

The iterative KM algorithm can be used to find the switch points and to compute the left and right centroid boundaries:

$$c_l = \frac{0.3077 \times -1.0 + 0.1429 \times 0.0 + 0.25 \times -0.8 + 0.25 \times 0.2}{0.3077 + 0.1429 + 0.25 + 0.25} \quad (16)$$

$$c_r = \frac{0.1429 \times -1.0 + 0.3077 \times 0.0 + 0.25 \times -0.8 + 0.25 \times 0.2}{0.1429 + 0.3077 + 0.25 + 0.25} \quad (17)$$

Finally, the interval centroid [c_l, c_r] can be calculated as $[-0.4815, -0.3081]$.

The equation for the left centroid boundary c_l expressed in (16) can be rewritten using the rule firing strengths as follows:

$$c_l = \frac{R_r^{21} \times u_{21} + R_l^{22} \times u_{22} + R^{31} \times u_{31} + R^{32} \times u_{32}}{R_r^{21} + R_l^{22} + R^{31} + R^{32}} \quad (18)$$

Next, the right and left firing levels of the used rules R^{21} and R^{22} can be expanded using the original membership grades:

$$R_r^{21} = f_{\tilde{e}2}^{r} \times f_{\dot{e}1}, \quad R_l^{22} = f_{\tilde{e}2}^{l} \times f_{\dot{e}2} \quad (19)$$

Both firing levels R_r^{21} and R_l^{22} are used to compute the left centroid boundary c_l. However, as shown in (19), R_r^{21} uses the upper membership grade $f_{\tilde{e}2}^{r}$ while R_l^{22} uses the lower membership grade $f_{\tilde{e}2}^{l}$ of the FOU of the input IT2 FS $\tilde{e}_2$. For the ET1 FSs presented in the previous section, no selected embedded T1 FSs from within the FOU can result in the interval bounds on the output of the IT2 FLS, because both the upper and the lower membership functions simultaneously contribute to the final results. Similar analysis has been recently presented in [30], where the adaptiveness and novelty properties of IT2 FLSs have been determined as the fundamental differences between IT2 and T1 FLSs.

This property can be seen as a strong advantage of IT2 FLS when analyzing their modeling capability as it was done in [25]. However, it should be noted that this behavior also causes the IT2 FLS to overestimate the range of the output uncertainty in specific scenarios. Hence, special care should be taken when the output centroid is interpreted as a measure of output uncertainty.

VI. Conclusion

This paper presented a study of the input-output uncertainty modeling capability of IT2 FLSs. First, the Monte Carlo simulation technique was used to simulate the effects of linguistic uncertainties and to compute the aggregated output result. This result was then compared to the interval centroid obtained with classical IT2 FLS. The IT2 FLS was found to overestimate the output uncertainty interval provided by the MC method.

To further investigate this behavior the concept of ET1 FSs was used to visualize the pair of embedded T1 FSs responsible for the interval centroid. The analysis revealed that the ET1 FSs frequently lie outside the FOU of the original IT2 FS and have membership grades greater than one. Finally, it was demonstrated that the cause for this phenomena is the simultaneous use of both the upper and lower membership grades of the input IT2 FSs when computing the interval centroid.

This property can be seen as a strong advantage of IT2 FLS when analyzing their modeling capability, however, it should be noted that this same property also causes the IT2 FLS to overestimate the range of the output uncertainty. In summary, a special attention should be paid to analyzing the interval output of the IT2 FLS when it is interpreted as a measure of output uncertainty.

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