Shadowed Type-2 Fuzzy Sets –Type-2 Fuzzy Sets with Shadowed Secondary Membership Functions

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Abstract—General Type-2 Fuzzy Sets (GT2 FSs) have been originally proposed to allow for modeling uncertainty associated with the membership grades of Type-1 (T1) FSs. However, because of the computational complexity associated with the processing of GT2 FSs, only their constrained version, the Interval T2 (IT2) FSs, have been widely used. While IT2 FSs allow for fast processing, they lack the expressive power of GT2 FSs when modeling various kinds of uncertainties. In order to combine the best of both types, this paper proposes a novel class of T2 FSs – the Shadowed Type-2 (ST2) FSs. The ST2 FS is a T2 FS with secondary membership functions represented as Shadowed Sets (SSs). Shadowed sets, originally proposed by Pedrycz, are directly induced by the T1 fuzzy membership functions and they are designed to conserve the amount of uncertainty in the original T1 FS. In a similar manner, an ST2 FS is directly induced by a GT2 FS via transforming all the T1 fuzzy secondary membership functions into Shadowed Sets. The resulting ST2 FSs can thus better capture the uncertainty in the original GT2 FSs when compared to the constrained IT2 FSs. Additionally, ST2 FSs offer very efficient computational framework since the secondary membership grades can only attain three values of 0, 1, or completely uncertain (shadowed) grade of [0,1]. This paper introduces the representation, the elementary set-theoretic operations and several methods for type-reduction and defuzzification of ST2 FSs. The modeling capability of ST2 SS was demonstrated on several examples.

Index Terms— **General Type-2 Fuzzy Sets, Interval Type-2 Fuzzy Sets, Shadowed Sets, Uncertainty Modeling**

I. Introduction

TYPE-2 Fuzzy Sets (T2 FSs) were originally proposed by Lofti Zadeh [1] to address the problem of over-specification of the real-valued membership degrees of Type-1 (T1) FSs. T2 FSs use membership degrees that are themselves FSs. Despite the powerful modeling capability of General T2 (GT2) FSs, the high computational complexity associated with processing and computing with GT2 FSs significantly hindered their practical use. Only very recently several novel representations of GT2 FSs such as geometric T2 FSs [2], [3] or the α*-planes* [4], [5] and the *zSlices* [6] representations allowed the emergence of novel applications of GT2 FSs [7]-[9].

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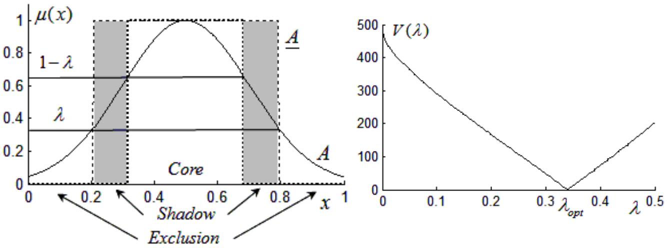
The computational overhead associated with computing with GT2 FSs led to a wide spread of applications of their constrained version – the Interval T2 (IT2) FSs [10], [11]. The IT2 FSs restrict the form of the secondary membership functions to intervals. This simplification allows for representing an IT2 FS using its Footprint Of Uncertainty (FOU). The FOU can be conveniently expressed using its upper and lower membership functions, which are T1 FSs. This fact allowed the development of efficient algorithms for processing IT2 FSs based on the interval arithmetic [12]. Many successful applications of IT2 FSs as well as IT2 Fuzzy Logic Systems (FLSs) can be found in literature [13]-[16]. Nevertheless, the imposed restriction of the interval secondary membership functions, where all secondary membership are only allowed to attain certain values of 0 or 1, can be seen as a significant limitation in situations where smoother ("fuzzier") representation of secondary uncertainty is required [7], [8].

To alleviate this issue and to combine the best of both worlds, a new class of T2 FSs is proposed in this paper – the Shadowed Type-2 (ST2) FSs. The ST2 FSs are T2 FSs with secondary membership functions represented as Shadowed Sets (SSs). The concept of SS was originally proposed by Pedrycz and it was developed to improve the observability and interpretability of T1 FSs and to alleviate the issues of excessive precision in describing imprecise concepts using T1 fuzzy membership functions [17]-[21]. An SS is directly induced by a T1 FS, which is divided into three regions of exclusion, core and shadow based on the T1 fuzzy membership grades. The optimal threshold value is automatically found by solving a simple optimization problem to conserve the overall amount of uncertainty modeled by the original T1 FS.

This paper outlines the novel concept of ST2 FSs. An ST2 FS is directly induced by a GT2 FS by transforming all the T1 fuzzy secondary membership functions into their SS forms. This transformation is performed by searching for an optimal pair of α*-planes*, which lead to a conservation of the amount of uncertainty modeled by the original GT2 FSs. Interestingly, the ST2 FSs can be completely described by a pair of IT2 FSs, the inner and the outer bounds. Hence, the set-theoretic operations and the type-reduction algorithms can be implemented using the established methods of IT2 FSs. The ST2 FSs offer improved uncertainty modeling, which is manifested in the structure of the type-reduced centroid that is itself represented as an SS. Three different defuzzification approaches are proposed, the optimistic, the pessimistic and the weighted defuzzification method. Finally, the construction



(a) (b)

Fig. 1 Shadowed set $\underline{A}$ induced by a Gaussian T1 FS A (a) and the optimization function $V(\lambda)$.

of ST2 FSs is demonstrated on several simple examples.

The rest of the paper is organized as follows. Section II reviews the concept of SSs. For the sake of completeness, the theory of GT2 FSs including the α-plane representation is discussed in Section III. Section IV introduces the proposed class of ST2 FSs. Several examples of ST2 FSs are presented in Section V and the paper is concluded in Section VI.

II. Shadowed Sets

The concept of Shadowed Sets was originally proposed by Pedrycz [17]-[21]. This concept alleviates the issues of excessive precision in describing imprecise concepts using T1 FSs. The central idea of SS is that people can easily assign membership values close to 0 or 1 but have significant difficulties assigning uncertain membership grades around the value 0.5. According to Pedrycz, the concept of SS was developed to improve the observability and interpretability of T1 FSs [17].

Consider a T1 FS A in the universe of discourse X defined using its membership function $\mu_A(x)$ for $x \in X$, as depicted in Fig 1(a). This T1 FS can be seen as a functional mapping between the input domain and the membership grade:

$$A : X \to [0,\, 1] \tag{1}$$

This T1 FS A induces a SS $\underline{A}$ [1].The process of obtaining $\underline{A}$ contains the reduction of the functional mapping in (1) into a three valued logic. In order to preserve the uncertainty encapsulated by the original T1 FS A, this process is achieved via elevating, reducing and balancing the fuzzy membership grades [17].

The SS $\underline{A}$ is induced by applying a threshold λ to the T1 fuzzy membership function $\mu_A(x)$. All membership grades above value $1-\lambda$ are elevated into a core with certain membership degree of 1. All membership grades below value λ are reduced into an exclusion region with certain membership degree of 0. All membership values between values λ and $1-\lambda$ are transformed into a completely uncertain shadow region with membership degree spanning the entire domain of [0, 1]. Hence, the SS $\underline{A}$ can be seen as a special case of a three-valued logic:

$$\underline{A} : X \to \{0, 1, [0,\, 1]\} \tag{2}$$

The thresholding operation that is applied during the construction of SSs does not require any user-specified threshold value. Instead, a suitable value of the threshold λ can be automatically obtained by solving a simple optimization problem. The basic idea is that the uncertainty retained in the shadow region should balance the uncertainty lost due to elevating and reducing the membership grades in the core and the exclusion regions. For a continuous domain, this optimization function $V(\lambda)$ can be specified as:

$$V(\lambda) = \left| \int_{x \in Excl} \mu(x)\, dx + \int_{x \in Core} (1 - \mu(x)\, dx) - \int_{x \in Sh} dx \right| \tag{3}$$

Here, the terms *Excl*, *Core* and *Sh* denote the regions of the primary domain X of the exclusion, core and shadow as denoted in Fig. 1(a). Note that $\lambda \in [0,\, 0.5]$ and that for the optimal value of λ_{opt} it holds that $V(\lambda_{opt}) = 0$. In practical implementations the fuzzy membership function $\mu(x)$ is typically discretized. Hence, the solution to (3) can be obtained as:

$$\lambda_i = \lambda_{opt} = \arg\min_{\lambda_i} V(\lambda_i) \tag{4}$$

The final SS $\underline{A}$ is depicted in Fig. 1(a) and Fig. 1(b) shows the function $V(\lambda)$ for various values of threshold λ.

III. General Type-2 Fuzzy Sets

This section first reviews the concept of GT2 FSs. Next, the recently developed α*-planes* representation for GT2 FSs is discussed.

A. General Type-2 Fuzzy Sets

A GT2 FS $\tilde{A}$ can be expressed on the universe of discourse X using its T2 fuzzy membership function $\mu_{\tilde{A}}(x,u)$, where $x \in X$ and $u \in J_x$ [10]:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u) / (x,u) \qquad J_x \subseteq [0,1] \tag{5}$$

Here, variable x and u are the primary and the secondary variables and J_x denotes the support of the secondary membership function also called the primary membership. Operator $\int\int$ denotes union over all possible values of x and u, and $\mu_{\tilde{A}}(x,u) \in [0,1]$.

By instantiating a specific value for $x = x'$, a vertical slice $\mu_{\tilde{A}}(x',u)$ of the fuzzy membership function $\mu_{\tilde{A}}(x,u)$ can be obtained. This vertical slice defines a secondary membership function $\mu_{\tilde{A}}(x = x',u)$ for $x' \in X$ and $\forall u \in J_{x'} \subseteq [0,1]$:

$$\mu_{\tilde{A}}(x = x',u) \equiv \mu_{\tilde{A}}(x') = \int_{u \in J_{x'}} f_{x'}(u) / u \qquad J_{x'} \subseteq [0,1] \tag{6}$$

[1] Because no consistent notation was found in the available literature, symbol $\underline{A}$ was introduced by the authors for denoting a Shadow Set.

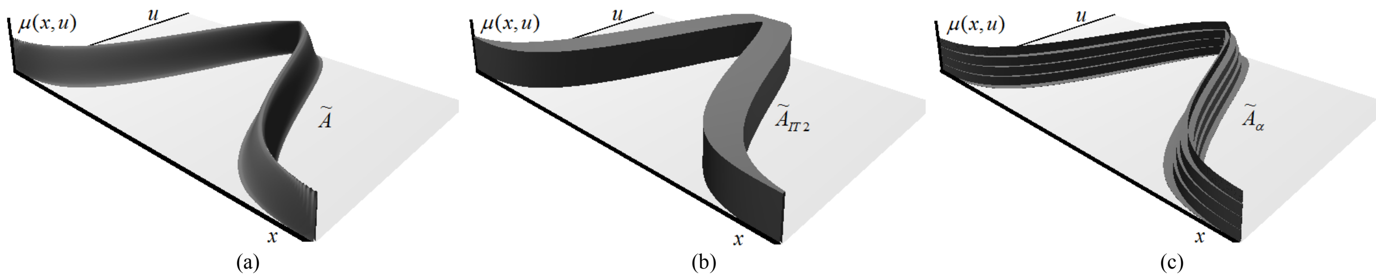


Fig. 2 General T2 fuzzy set $\widetilde{A}$ (a), its IT2 variant $\widetilde{A}_{IT2}$ (b) and its α -plane representation $\widetilde{A}_{\alpha}$ (c).

Here, $f_{x'}(u)$ denotes the secondary grade or the amplitude of the secondary membership function and $f_{x'}(u) \in [0,1]$. Assuming that the domain of primary variable x is discretized using N samples, the GT2 FS $\widetilde{A}$ can be represented as a composition of all its vertical slices:

$$\widetilde{A} = \sum_{i=1}^{N} \left[\int_{u \in J_{x_i}} f_{x_i}(u)/u \right] \Big/ x_i \tag{7}$$

B. α -Planes Representation of GT2 FSs

The following notation for the α *-plane* representation was adopted from [4], [5]. The α *-plane* representation constitutes a horizontal slice representation for GT2 FSs that was independently developed by several authors [4]-[6], [23].

An α *-plane* $\widetilde{A}_{\alpha}$ of a GT2 Fs $\widetilde{A}$ can be defined as the union of all primary memberships of $\widetilde{A}$ with secondary membership grades that are greater or equal to α :

$$\widetilde{A}_{\alpha} = \int_{\forall x \in X} \int_{\forall u \in J_x} \{(x,u) \,|\, f_x(u) \geq \alpha\} \tag{8}$$

An α *-cut* of the secondary membership function $\mu_{\widetilde{A}}(x)$ can be denoted as $S_{\widetilde{A}}(x|\alpha)$ and expressed as:

$$S_{\widetilde{A}}(x|\alpha) = [s_L(x|\alpha), s_R(x|\alpha)] \tag{9}$$

Hence, an α *-plane* $\widetilde{A}_{\alpha}$ can be seen as a composition of all individual α *-cuts* of all of its secondary membership functions:

$$\widetilde{A}_{\alpha} = \int_{\forall x \in X} S_{\widetilde{A}}(x|\alpha) = \int_{\forall x \in X} \left(\int_{\forall u \in [s_L(x|\alpha), s_R(x|\alpha)]} u \right) \Big/ x \tag{10}$$

It is apparent that the FOU of the GT2 FS $\widetilde{A}$ can now be denoted as:

$$FOU(\widetilde{A}) = \widetilde{A}_0 \tag{11}$$

Each α *-plane* $\widetilde{A}_{\alpha}$ is bounded from above by its upper membership function $\overline{\mu}_{\widetilde{A}}(x|\alpha)$ and from below by its lower membership function $\underline{\mu}_{\widetilde{A}}(x|\alpha)$. Using the α *-cuts* boundaries of each vertical slice (9), these bounds can be expressed as:

$$\overline{\mu}_{\widetilde{A}}(x|\alpha) = \int_{\forall x \in X} s_R(x|\alpha) \tag{12}$$

$$\underline{\mu}_{\widetilde{A}}(x|\alpha) = \int_{\forall x \in X} s_L(x|\alpha) \tag{13}$$

By raising the α *-plane* $\widetilde{A}_{\alpha}$ to the level of α , a special IT2 FS is created. This FS was called α *-level* T2 FS $R_{\widetilde{A}_{\alpha}}(x,u)$ in [23] and expressed as follows:

$$R_{\widetilde{A}_{\alpha}}(x,u) = \alpha / \widetilde{A}_{\alpha}, \ \forall x \in X, \ \forall u \in J_x \tag{14}$$

Finally, according to Liu's representation theorem, the GT2 FS $\widetilde{A}$ can be constructed as a composition of all of its individual α *-level* T2 FSs [4]:

$$\widetilde{A} = \bigcup_{\alpha \in [0,1]} \alpha \big/ \widetilde{A}_{\alpha} \tag{15}$$

It should be noted here, that the $\bigcup$ sign denotes the union set-theoretic operations, which for each point computes the maximum membership grade for all α *-planes*. For comparison, a GT2 FSs, an IT2 FSs and a GT2 FSs represented using three α *-planes* are depicted in Fig. 2.

IV. Shadowed Type-2 Fuzzy Sets

This section introduces the new class of T2 FS – the Shadowed T2 FSs. First, the representation of ST2 FSs is considered followed by description of the elementary set theoretic operations. Finally, the important operations of type-reduction and defuzzification are defined.

A. Assumption of Convex Secondary Membership Functions

In the rest of this paper it is assumed that all secondary membership functions $f_x(u)$ of the GT2 FS $\widetilde{A}$ are convex T1 FSs with the following properties:

$$f_x(u) = \begin{cases} g_x(u) & u \in [s_L(x|0), s_L(x|1)], \quad g_x(u) \in [0,1] \\ 1 & u \in [s_L(x|1), s_R(x|1)] \\ h_x(u) & u \in [s_R(x|1), s_R(x|0)], \quad h_x(u) \in [0,1] \\ 0 & Otherwise \end{cases} \tag{16}$$

where $g_x(u)$ and $h_x(u)$ are monotonically non-decreasing and monotonically non-increasing functions in their respective domains.

This assumption on the nature of secondary membership does not impose a major limitation to the applicability of the proposed concept of ST2 FSs. Examples of convex T1 FS FSs are triangular, trapezoidal and Gaussian T1 FSs, which are the most commonly used types of secondary membership functions for GT2 FSs.

B. Representation of Shadowed T2 FSs

An ST2 FS $\underline{\tilde{A}}$ is induced by a GT2 FS $\tilde{A}$. The process of constructing $\underline{\tilde{A}}$ constraints all the secondary membership functions of $\tilde{A}$ to be SSs. By doing so, the computational complexity associated with working with GT2 FSs is significantly reduced because ST2 FS can take advantage of the efficient algorithms of IT2 FSs. At the same time, the ST2 FSs offer improved description of uncertainty, which is captured using the SSs rather than simple interval values for the secondary membership functions.

The ST2 FS $\underline{\tilde{A}}$ can be seen as functional mapping:

$$\underline{\tilde{A}} : X \times [0,1] \to \{0,1,[0,\ 1]\} \tag{17}$$

Its membership function can be expressed as follows:

$$\underline{\tilde{A}} = \{(x,u),\ \mu_{\underline{\tilde{A}}}(x,u)\ \big|\ x \in X,\ u \in [0,1],\ \mu_{\underline{\tilde{A}}}(x,u) \in \{0,1,[0,\ 1]\}\} \tag{18}$$

Similarly to the T1 FSs case, the process of constructing an ST2 FS $\underline{\tilde{A}}$ based on a GT2 FS $\tilde{A}$ includes elevation, reduction and balancing of the membership grades. Recall, that an SS $\underline{A}$ was constructed from a T1 FS A using threshold λ, where all membership grades below value λ and above value $1-\lambda$ are reduced or elevated into an exclusion and core regions and all membership values between values λ and $1-\lambda$ are transformed into a maximally uncertain shadow region with membership degree spanning the entire domain of [0, 1]. Hence, the process of computing the optimal value of threshold λ consisted of finding such levels of λ and $1-\lambda$ of the α*-cuts* that would minimize cost function $V(\lambda)$ expressed in (3).

The ST2 FS $\underline{\tilde{A}}$ is constructed using a suitable threshold $\tilde{\lambda}$.

The core $core(\underline{\tilde{A}})$ of ST2 FS $\underline{\tilde{A}}$ can be described as a footprint of $\tilde{A}$ where all secondary membership degrees are greater than $1-\tilde{\lambda}$.

$$core(\underline{\tilde{A}}) = \{x,u \mid \mu_{\tilde{A}}(x,u), x \in X, u \in [0,1], \mu_{\tilde{A}}(x,u) > (1-\tilde{\lambda})\} \tag{19}$$

The exclusion region $excl(\underline{\tilde{A}})$ of ST2 FS $\underline{\tilde{A}}$ can be defined as a footprint of $\tilde{A}$ where all secondary memberships are less than threshold $\tilde{\lambda}$:

$$excl(\underline{\tilde{A}}) = \{x,u \mid \mu_{\tilde{A}}(x,u), x \in X, u \in [0,1], \mu_{\tilde{A}}(x,u) < \tilde{\lambda}\} \tag{20}$$

The shadow region $sh(\underline{\tilde{A}})$ of ST2 FS $\underline{\tilde{A}}$ can be constructed as a footprint of $\tilde{A}$ where all secondary memberships are between thresholds values $\tilde{\lambda}$ and $1-\tilde{\lambda}$:

$$sh(\underline{\tilde{A}}) = \{x,u \mid \mu_{\tilde{A}}(x,u), x \in X, u \in [0,1], \tilde{\lambda} \le \mu_{\tilde{A}}(x,u) \le (1-\tilde{\lambda})\} \tag{21}$$

The process of locating the optimal value of threshold $\tilde{\lambda}$ is automated and does not require any user input. It consists of finding a pair of α*-planes* at levels $\tilde{\lambda}$ and $1-\tilde{\lambda}$, which optimize a fitness function $V(\tilde{\lambda})$ similar to the one in (3). The objective function is composed of three components, which express the amount of uncertainty in regions that were reduced ($V^R(\tilde{\lambda})$), elevated ($V^E(\tilde{\lambda})$) or balanced ($V^B(\tilde{\lambda})$). Based on the notation depicted in Fig. 3(a) the reduction component can be expressed as:

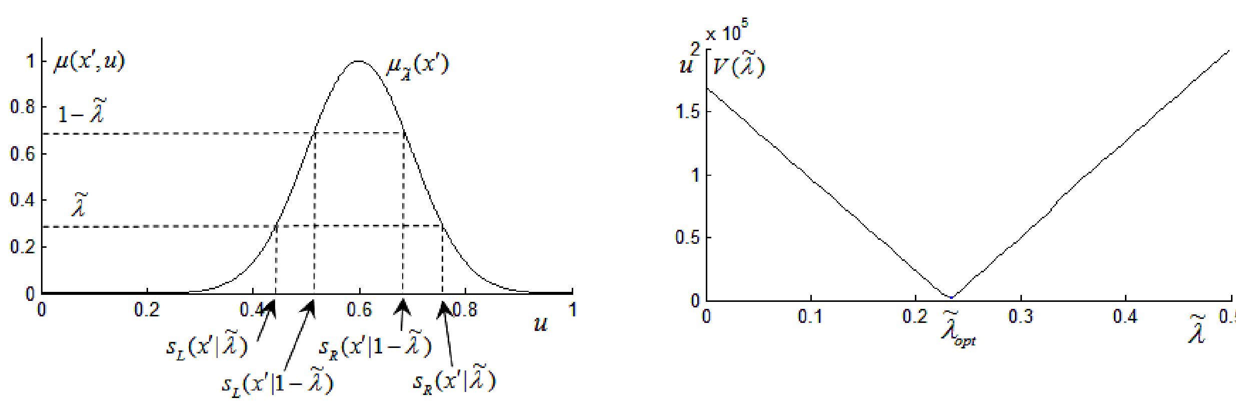
$$V^R(\tilde{\lambda}) = \int_{x\in X}\int_0^{s_L(x|\tilde{\lambda})} \mu_{\tilde{A}}(x,u)\,du\,dx + \int_{x\in X}\int_{s_R(x|\tilde{\lambda})}^{1} \mu_{\tilde{A}}(x,u)\,du\,dx \tag{22}$$

The elevation component is then calculated as:

$$V^E(\tilde{\lambda}) = \int_{x\in X}\int_{s_L(x|1-\tilde{\lambda})}^{s_R(x|1-\tilde{\lambda})} \mu_{\tilde{A}}(x,u)\,du\,dx \tag{23}$$

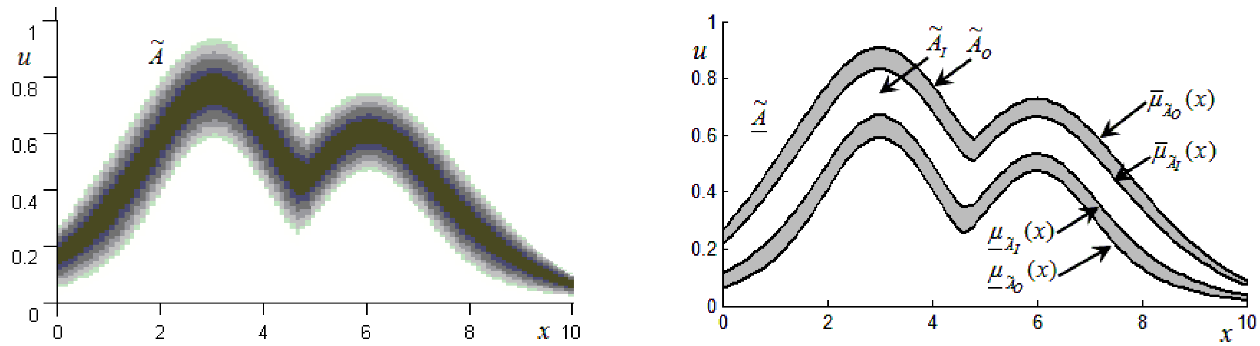
Finally, the balance component can be obtained as:

$$V^B(\tilde{\lambda}) = \int_{x\in X}\int_{s_L(x|\tilde{\lambda})}^{s_L(x|1-\tilde{\lambda})} du\,dx + \int_{x\in X}\int_{s_R(x|1-\tilde{\lambda})}^{s_R(x|\tilde{\lambda})} du\,dx \tag{24}$$



(a) (b)

Fig. 3 Secondary membership function of GT2 FS $\tilde{A}$ and its segmentation using two selected α*-planes* (b) and the optimization function $V(\tilde{\lambda})$ (b).



(a) (b)

Fig. 4 GT2 FS $\widetilde{A}$ (a) and its induced ST2 FS $\underset{\sim}{A}$ (b).

By combining all three components the optimization function $V(\widetilde{\lambda})$ can be constructed as:

$$V(\widetilde{\lambda}) = \left| V^R(\widetilde{\lambda}) + V^E(\widetilde{\lambda}) - V^B(\widetilde{\lambda}) \right| \tag{25}$$

For the continuous case, the optimal value of the threshold $\widetilde{\lambda}_{opt}$ would result in $V(\widetilde{\lambda}_{opt}) = 0$. However, in practical implementations when the domains of the primary and secondary variables are discretized and the GT2 FS $\widetilde{A}$ is represented in the α*-plane* framework with a finite number of α*-planes* the solution can be obtained as:

$$\widetilde{\lambda}_i = \widetilde{\lambda}_{opt} = \underset{\widetilde{\lambda}_i}{\arg\min}\, V(\widetilde{\lambda}_i) \tag{26}$$

An example of the optimization function $V(\widetilde{\lambda})$ is depicted in Fig. 3(b).

An ST2 FS $\underset{\sim}{A}$ induced by a GT2 FS $\widetilde{A}$ with Gaussian secondary membership functions is depicted in Fig. 4. As it can be clearly seen in the figure, for the GT2 FS with convex secondary membership functions (e.g. triangular, trapezoidal or Gaussian) the ST2 FS $\underset{\sim}{A}$ can be completely described using its inner and outer boundaries $\widetilde{A}_I$ and $\widetilde{A}_O$. Each boundary is composed of two T1 fuzzy membership functions, the lower ($\underline{\mu}_{\widetilde{A}_I}(x)$, $\underline{\mu}_{\widetilde{A}_O}(x)$) and the upper ($\bar{\mu}_{\widetilde{A}_I}(x)$, $\bar{\mu}_{\widetilde{A}_O}(x)$) membership functions as depicted in Fig. 4(b). The outer boundary marks the boundary between the exclusion and the shadow region. Similarly, the inner boundary marks the transition from the shadow to the core region.

This simplified view offers a convenient way to fully describe the ST2 FS $\underset{\sim}{A}$ as:

$$\underset{\sim}{A} = \{\widetilde{A}_I, \widetilde{A}_O\} \tag{27}$$

Here, both $\widetilde{A}_I$ and $\widetilde{A}_O$ are IT2 FSs. As shown in the remainder of this section, all operations with ST2 FSs can be performed solely using this simplified representation with the help of the computationally efficient interval arithmetic. It is interesting to note that the inner boundary $\widetilde{A}_I$ and the outer boundary $\widetilde{A}_O$ can be directly obtained as the boundaries of two α*-planes* at levels $\widetilde{\lambda}$ and $1-\widetilde{\lambda}$.

Property 1: *Inclusion of Boundaries*:

$$\widetilde{A}_I \subseteq \widetilde{A}_O \tag{28}$$

The proof of **Property 1** is trivial and it is based on the containment of α*-planes* property and the fact that by definition $\widetilde{\lambda} \le (1-\widetilde{\lambda})$.

C. Set theoretic operations with ST2 FSs

Here, the three elementary operations of intersection, union and complement on ST2 FSs are defined. Recall that for two IT2 FSs $\widetilde{A}$ and $\widetilde{B}$ the intersection (also known as the meet) operation is performed as follows [10]:

$$\widetilde{A} \cap \widetilde{B} = 1 \Big/ [\underline{\mu}_{\widetilde{A}}(x) \wedge \underline{\mu}_{\widetilde{B}}(x), \bar{\mu}_{\widetilde{A}}(x) \wedge \bar{\mu}_{\widetilde{B}}(x)] \quad \forall x \in X \tag{29}$$

Here, symbol $\wedge$ denotes a t-norm operation, e.g. the minimum or product. The intersection of two ST2 FSs $\underset{\sim}{A}$ and $\underset{\sim}{B}$ (depicted in Fig. 5(a)) can be defined as follows:

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \{\widetilde{A}_I, \widetilde{A}_O\} \cap \{\widetilde{B}_I, \widetilde{B}_O\} = \{\widetilde{A}_I \cap \widetilde{B}_I, \widetilde{A}_O \cap \widetilde{B}_O\} \tag{30}$$

The method for intersection of two IT2 FSs described in (29) can be used to calculate individual components in (30). An example of the result of the meet operation on two ST2 FS $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is depicted in Fig. 5(b).

Next, the union (also known as the join) operation of two IT2 FSs can be performed as follows [10]:

$$\widetilde{A} \cup \widetilde{B} = 1 \Big/ [\underline{\mu}_{\widetilde{A}}(x) \vee \underline{\mu}_{\widetilde{B}}(x), \bar{\mu}_{\widetilde{A}}(x) \vee \bar{\mu}_{\widetilde{B}}(x)] \quad \forall x \in X \tag{31}$$

Here, symbol $\vee$ denotes a t-conorm operation, e.g. the maximum. The union of two ST2 FSs $\underset{\sim}{A}$ and $\underset{\sim}{B}$ can be defined as follows:

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \{\widetilde{A}_I, \widetilde{A}_O\} \cup \{\widetilde{B}_I, \widetilde{B}_O\} = \{\widetilde{A}_I \cup \widetilde{B}_I, \widetilde{A}_O \cup \widetilde{B}_O\} \tag{32}$$

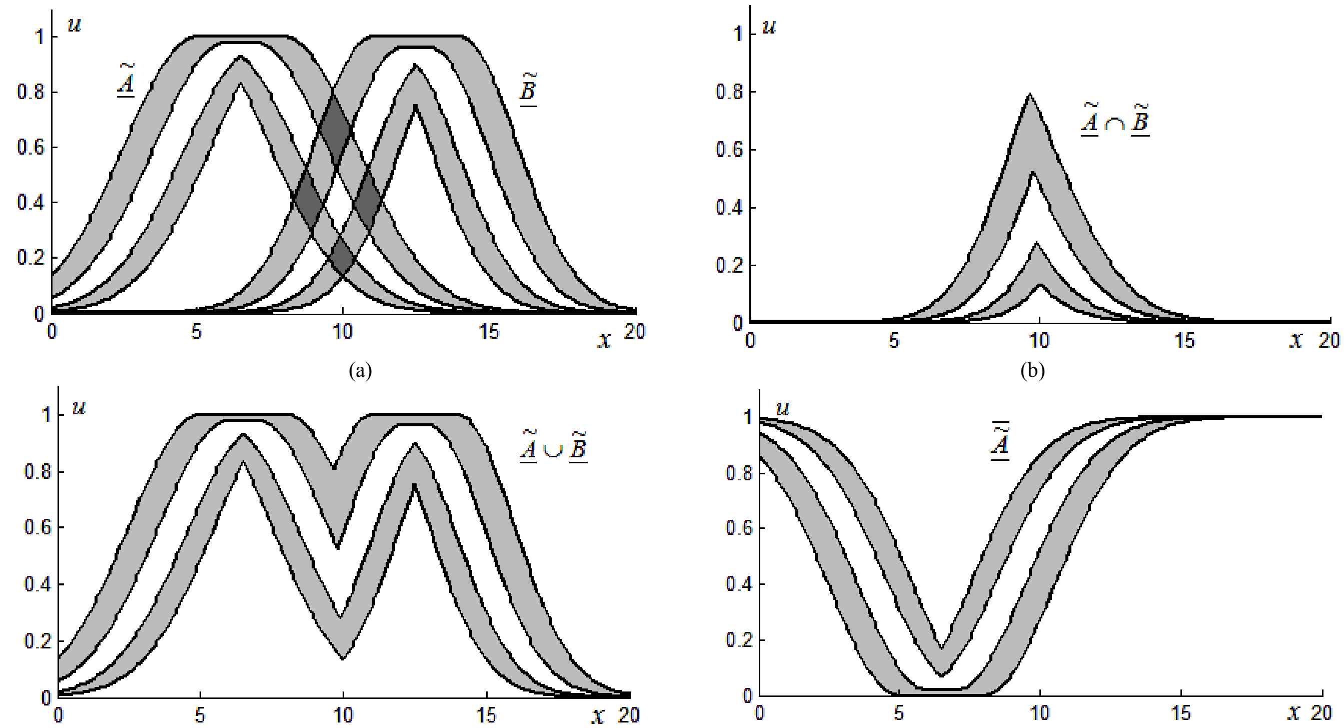


Fig. 5 Two ST2 FSs $\tilde{\underline{A}}$ and $\tilde{\underline{B}}$ (a), the results of the set theoretic operations of meet $\tilde{\underline{A}} \cap \tilde{\underline{B}}$ (b), join $\tilde{\underline{A}} \cup \tilde{\underline{B}}$ (c) and complement $\overline{\tilde{\underline{A}}}$ (d).

The method for union of two IT2 FSs described in (31) can be used to calculate individual components in (32). An example of the results of the join operation on two ST2 FS $\tilde{\underline{A}}$ and $\tilde{\underline{B}}$ is depicted in Fig. 5(c).

Finally, the complement of an IT2 FSs $\overline{\tilde{A}}$ can be computed as follows [10]:

$$\overline{\tilde{A}} = 1/[1-\bar{\mu}_{\tilde{A}}(x), 1-\underline{\mu}_{\tilde{A}}(x)] \quad \forall x \in X \tag{33}$$

The complement of a ST2 FSs $\overline{\tilde{\underline{A}}}$ can then be obtained as follows:

$$\overline{\tilde{\underline{A}}} = \overline{\{\tilde{A}_I, \tilde{A}_O\}} = \{\overline{\tilde{A}}_I, \overline{\tilde{A}}_O\} \quad \forall x \in X \tag{34}$$

The method for computing the complement an IT2 FS provided in (33) can be used to calculate individual components in (34). An example of the results of the complement operation of ST2 FS $\tilde{\underline{A}}$ is depicted in Fig. 5(d).

D. Type-reduction of ST2 FSs

Similarly to the basic set theoretic operations, the type-reduction of ST2 FSs also takes advantage of the well-established and computationally efficient algorithms of IT2 FSs. Recall that the interval centroid $C_{\tilde{A}}$ of an IT2 FS $\tilde{A}$ can be fully described using its left and right boundaries $[c_l, c_r]$, which can be computed as follows [12]:

$$c_l = \frac{\sum_{i=1}^{L} x_i \bar{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{L} \bar{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{A}}(x_i)} \tag{35}$$

$$c_r = \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} x_i \bar{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} \bar{\mu}_{\tilde{A}}(x_i)} \tag{36}$$

The switch points L and R can be calculated using one of the available algorithms, for example the Enhanced Karnik Mendel (EKM) iterative algorithm [24].

The centroid of an ST2 FS $\tilde{\underline{A}}$ denoted as $C_{\tilde{\underline{A}}}$ can be described using two interval T1 FSs describing the inner and the outer centroids $C^I_{\tilde{\underline{A}}}$ and $C^O_{\tilde{\underline{A}}}$:

$$C_{\tilde{\underline{A}}} = \{C^I_{\tilde{\underline{A}}}, C^O_{\tilde{\underline{A}}}\} \tag{37}$$

The inner and the outer centroid can be computed by independently type-reducing the inner and the outer boundary sets $\tilde{A}_I$ and $\tilde{A}_O$. Hence:

$$C_{\tilde{\underline{A}}} = \{C_{\tilde{A}_I}, C_{\tilde{A}_O}\} = \{c^I_l, c^I_r, c^O_l, c^O_r\} \tag{38}$$

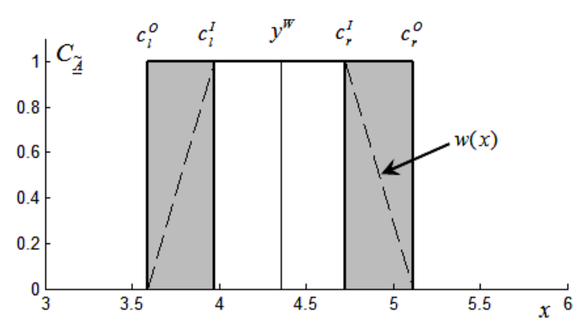


Fig. 6 Centroid of the ST2 FS $\tilde{\underline{A}}$ from Fig. 3(b).

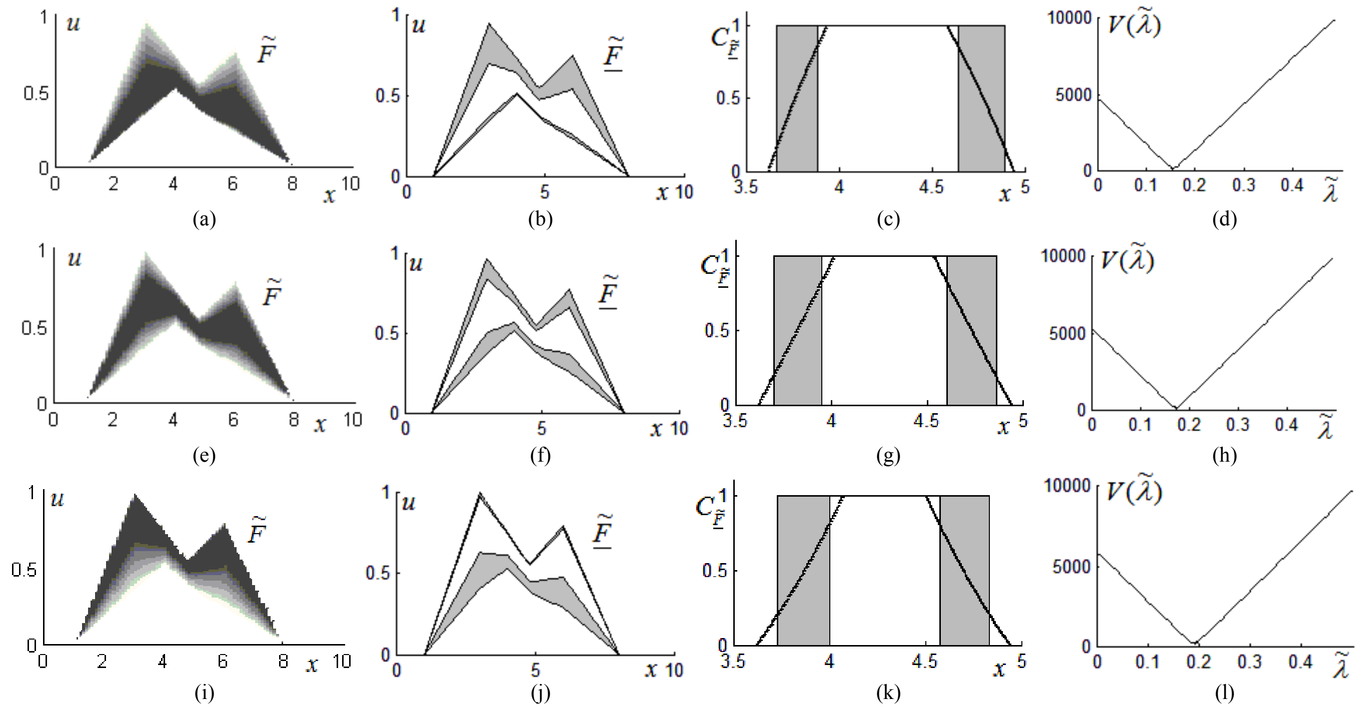


Fig. 7 GT2 FS $\tilde{F}$ (a), (e), (i), its induced ST2 FS $\underline{\tilde{F}}$ (b), (f), (j), the type-reduced centroid $C_{\underline{\tilde{F}}}$ (c), (g), (k) and the optimized function $V(\tilde{\lambda})$ (d), (h), (l) for GT2 FS $\tilde{F}$ with parameter $w = \{0.1, 0.5, 0.9\}$.

The outer centroid $C^O_{\underline{\tilde{A}}}$ marks the boundary between the exclusion region and the shadowed region of the centroid. Similarly, the inner centroid $C^I_{\underline{\tilde{A}}}$ creates a boundary between the shadowed boundary and the core region. The centroid of the ST2 FSs is depicted in Fig. 6.

E. Defuzzification of ST2 FSs

One of the advantages of ST2 FSs is the improved description of uncertainty, when compared to IT2 FSs. This fact can be apparent when inspecting the centroid of the ST2 FS depicted in Fig. 6, which is composed of core, shadow and exclusion regions. This more complex centroid structure allows for improved modeling of various uncertainties. In this paper, three possible defuzzification methods are proposed, namely the optimistic, the pessimistic and the weighted defuzzification methods.

The optimistic defuzzification method is optimistic in the sense that the output value is optimistically assumed to be located in the core region. Hence, the system is optimistic that the amount of present uncertainty is low. The output value y^O produced by the optimistic defuzzifier can be computed as:

$$y^O = \frac{c^I_l + c^I_r}{2} \tag{39}$$

The pessimistic defuzzification method pessimistically assumes the largest amount of uncertainty. Hence, the output value y^P is expected to be located anywhere within the shadowed and the core region:

$$y^P = \frac{c^O_l + c^O_r}{2} \tag{40}$$

Finally, the weighted defuzzification computes the output value y^W by combining the pessimistic and optimistic defuzzification method through a weighted process. Different weighting functions could be applied. The trapezoidal weighting function $w(x)$ is proposed here as an example:

$$w(x) = \begin{cases} 0 & x < c^O_l \\ \dfrac{x - c^O_l}{c^I_l - c^O_l} & c^O_l \le x < c^I_l \\ 1 & c^I_l \le x < c^I_R \\ \dfrac{c^O_r - x}{c^O_r - c^I_r} & c^I_r \le x < c^O_r \\ 0 & c^O_r \le x \end{cases} \tag{41}$$

The defuzzification output can then be computed as follows:

$$y^W = \frac{\sum_{i=1}^{N} w(x_i) x_i}{\sum_{i=1}^{N} w(x_i)} \tag{42}$$

The trapezoidal weighting function $w(x)$ and the final defuzzified value for the weighted defuzzification method are depicted in Fig. 6.

V. Examples of Shadowed Type-2 Fuzzy Sets

This Section demonstrates the construction of ST2 FSs from a GT2 FSs with different shapes of secondary membership functions. An example of ST2 FS induced by a GT2 FS with Gaussian secondary membership functions was previously shown in Fig. 4. In this Section a trapezoidal secondary membership functions are considered.

Consider a GT2 FS $\widetilde{F}$ with piecewise linear FOU, which can be defined in terms of its lower and upper membership functions as follows:

$$\overline{\mu}_{\widetilde{F}}(x) = \max\left\{\begin{bmatrix} (x-1), & x \in \langle 1,3\rangle \\ (7-x)/4, & x \in \langle 3,7\rangle \\ 0, & x \notin \langle 1,7\rangle \end{bmatrix}, \begin{bmatrix} (x-2)/5, & x \in \langle 2,6\rangle \\ (16-2x)/5, & x \in \langle 6,8\rangle \\ 0, & x \notin \langle 2,8\rangle \end{bmatrix}\right\} \tag{43}$$

$$\underline{\mu}_{\widetilde{F}}(x) = \max\left\{\begin{bmatrix} (x-1)/6, & x \in \langle 1,4\rangle \\ (7-x)/6, & x \in \langle 4,7\rangle \\ 0, & x \notin \langle 1,7\rangle \end{bmatrix}, \begin{bmatrix} (x-3)/6, & x \in \langle 3,5\rangle \\ (8-x)/9, & x \in \langle 5,8\rangle \\ 0, & x \notin \langle 3,8\rangle \end{bmatrix}\right\} \tag{44}$$

The GT2 FS $\widetilde{F}$ maintains trapezoidal secondary membership functions. The position of the left and the right boundary points of the center interval of the trapezoid can be adjusted using parameter *w* as:

$$center_L(x) = \underline{\mu}_{\widetilde{F}}(x) + 0.6\,w\,(\overline{\mu}_{\widetilde{F}}(x) - \underline{\mu}_{\widetilde{F}}(x)) \tag{45}$$

$$center_R(x) = \overline{\mu}_{\widetilde{F}}(x) - 0.6(1-w)\,(\overline{\mu}_{\widetilde{F}}(x) - \underline{\mu}_{\widetilde{F}}(x)) \tag{46}$$

Fig. 7(a), 7(e) and 7(i) depict the GT2 FSs $\widetilde{F}$ constructed with the value of parameter $w = \{0.1, 0.5, 0.9\}$. The three induced ST2 FSs $\underline{\widetilde{F}}$ are depicted in Fig. 7(b), 7(f), 7(j). It can be observed how the shapes of the secondary membership functions affect the locations of the shadow and the core regions.

The type-reduced centroids of the ST2 FSs $\underline{\widetilde{F}}$ are depicted in Fig. 7(c), 7(g), 7(k). For a comparison, the T1 centroid of the original GT2 FSs $\widetilde{F}$ is depicted by thick line in the figures. It can be observed that the location of the shadowed region corresponds to the location of the uncertain membership grades of the T1 fuzzy centroid around the value of 0.5.

Finally, Fig. 7(d), 7(h), 7(l) show the value of the optimization function $V(\widetilde{\lambda})$. For the construction of the depicted figures, 100 α *-planes* were used for the original GT2 FSs. The optimal threshold values $\widetilde{\lambda}$ for the three different values of parameter *w* were computed as follows 0.1531, 0.1735 and 0.1939, respectively.

VI. CONCLUSION

This paper proposed a novel class of T2 FSs – the Shadowed Type-2 Fuzzy Sets. The ST2 FS is a T2 FS with secondary membership functions represented as Shadowed Sets. The concept of ST2 FSs combines improved uncertainty modeling capability of GT2 FSs with the computational effectiveness of IT2 FSs. The representation of ST2 FSs was introduced together with the basic set-theoretic operations and methods for type-reduction and defuzzification. Future work will be focused on developing the theory of fuzzy logic systems and fuzzy inference based on ST2 FSs and on applying to the ST2 FSs to real world problems.

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