Centroid Density of Interval Type-2 Fuzzy Sets: Comparing Stochastic and Deterministic Defuzzification

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Abstract— **Recently, Type-2 (T2) Fuzzy Logic Systems (FLSs) gained increased attention due to their capability to better describe, model and cope with the ubiquitous dynamic uncertainties in many engineering applications. By far the most widely used type of T2 FLSs are the Interval T2 (IT2) FLSs. This paper provides a comparative analysis of two fundamentally different approaches to defuzzification of IT2 Fuzzy Sets (FSs) - the deterministic Karnik-Mendel Iterative Procedure (KMIP) and the stochastic sampling defuzzifier. As previously demonstrated by other researchers, these defuzzification algorithms do not always compute identical output values. In the presented work, the concept of centroid density of an IT2 FS is introduced in order to explain such discrepancies. It was demonstrated that the stochastic sampling defuzzification method converges towards the center of gravity of the proposed centroid density function. On the other hand, the KMIP method calculates the midpoint of the interval centroid obtained according to the extension principle. Since the information about the centroid density is removed via application of the extension principle, the two methods produce inevitably different results. As further demonstrated, this difference significantly increases in case of non-symmetric IT2 FSs.**

Keywords-Interval Type-2 Fuzzy Sets; Defuzzification; Centroid; Karnik-Mendel Algorithms; Sampling Defuzzifier

I. INTRODUCTION

Type-2 Fuzzy Logic Systems (T2 FLSs), originally proposed by Zadeh [1], constitute powerful tool for dealing with dynamic uncertainty. The T2 Fuzzy Sets (FSs) introduce additional design dimension for modeling and describing the uncertainty in the specific problem domain [2]. Unlike the Type-1 (T1) FSs with fixed membership functions, the T2 fuzzy membership functions are defined using the Footprint-of-Uncertainty (FOU) offering more design degrees of freedom [3]. The defuzzification of T2 FSs contains a type-reduction phase, which acts as a mapping between the original T2 FS and the type-reduced T1 FS, also called the centroid of the T2 FS [2], [3]. Despite some new recently introduced representations of general T2 FSs such as geometric T2 fuzzy sets [4], *zSlices* [5], α - planes [6], [7] or α - cuts [8], the Interval T2 (IT2) FSs are still most commonly used [9]. IT2 FLSs have been successfully applied in a wide range of applications [10]-[15].

This paper provides a comparative analysis of two fundamentally different approaches to defuzzification of IT2 FSs [9], the Karnik-Mendel Iterative Procedure (KMIP) [16] and the sampling defuzzification method [17]. The KMIP is a deterministic approach, which concentrates on bounding the type-reduced centroid of the IT2 FS by its left-most and right-most points [18]. The sampling defuzzification method, proposed by Greenfield et al., uses stochastic sampling method to sample the set of all embedded fuzzy sets and provide an estimate of the true defuzzified value [19]. The sampling defuzzification method was developed as a cut-down version of the original exhaustive defuzzification method, which enumerates and defuzzifies all available embedded fuzzy sets [20]. Other available methods for defuzzification of IT2 FSs are the direct defuzzifier [21], the Nie-Tan method [22], or the collapsing method [23].

As demonstrated by other researchers, the KMIP method and the sampling defuzzification method do not always compute identical solutions [24]. In this paper, the concept of centroid density of an IT2 FS is introduced to provide an experimental explanation for such discrepancies. The concept of centroid density describes the histogram of defuzzified embedded fuzzy sets within the boundaries of the type-reduced centroid. It is shown through experimental study that the stochastic sampling defuzzification method converges towards the center of gravity of the proposed centroid density function. On the other hand, the KMIP method computes the midpoint of the interval centroid obtained according to Zadeh's extension principle. Because the information about the centroid density is removed via application of the extension principle, the two methods produce inevitably different results. This difference becomes especially significant in case of non-symmetric IT2 FSs.

The rest of the paper is organized as follows. Section II provides an overview of IT2 FSs together with a review of the considered defuzzification techniques. Section III introduced the concept of centroid density for an IT2 FS. The comparative analysis is presented in Section IV and the paper is concluded in Section V.

II. Interval Type-2 Fuzzy Sets

This section reviews fundamentals about the IT2 fuzzy sets together with a brief description of the defuzzification techniques considered in this work.

A. Interval Type-2 Fuzzy Sets

The IT2 FSs were introduced as a simplification to the general T2 FSs, which have been rarely applied to engineering problems due to their immense computational complexity. An IT2 FS $\widetilde{A}$ can be described as:

$$\widetilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u) \qquad J_x \subseteq [0,1] \tag{1}$$

Here, x and u constitute the primary and the secondary variables and J_x is the primary membership of variable x. The secondary grades of IT2 FS are all limited to 1. Hence, the IT2 FS can be completely described by its Footprint of Uncertainty (FOU), which is the area of non-zero secondary grade.

The vertical slice of an IT2 fuzzy set defines the secondary membership function. It can be obtained by instantiating the primary variable x into a specific value x':

$$\mu_{\widetilde{A}}(x = x', u) \equiv \mu_{\widetilde{A}}(x') = \int_{u\in J_{x'}} 1/u \quad J_{x'} \subseteq [0,1] \tag{2}$$

Using the concept of vertical slices the FOU can be expressed as follows:

$$FOU(\widetilde{A}) = \bigcup_{x\in X} J_x \tag{3}$$

According the Mendel-John representation theorem [25], the IT2 FS can be also seen as a collection of all its embedded fuzzy sets. An embedded fuzzy $\widetilde{A}_e$ can be described as:

$$\widetilde{A}_e = \sum_{i=1}^{N} [1/\theta_i]/x_i \quad \theta_i \in J_{x_i} \subseteq U \in [0,1] \tag{4}$$

Using this concept the IT2 FS $\widetilde{A}$ can be also expressed as:

$$\widetilde{A} = \bigcup_{j=1}^{n} \widetilde{A}_e^j \tag{5}$$

Alternatively, the FOU of an IT2 fuzzy set $\widetilde{A}$ can be conveniently and completely described by its upper and lower membership functions $\underline{\mu}(x)$ and $\overline{\mu}(x)$:

$$FOU(\widetilde{A}) = \bigcup_{\forall x\in X} (\underline{\mu}_{\widetilde{A}}(x),\ \overline{\mu}_{\widetilde{A}}(x)) \tag{6}$$

During the output processing stage the output IT2 FS $\widetilde{B}$ must be first type-reduced, which results in the centroid of the IT2 FS $C(\widetilde{B})$ [20]:

$$C(\widetilde{B}) = \int_{\theta_1\in J_{x_1}} \cdots \int_{\theta_N\in J_{x_N}} 1 \Bigg/ \frac{\sum_{i=1}^{N} x_i\theta_i}{\sum_{i=1}^{N}\theta_i} \tag{7}$$

This formula was derived using Zadeh's extension principle [28]. In the special case of IT2 FSs the centroid $C(\widetilde{B})$ is an interval T1 FS, which can be described by its left and right boundary points y_L and y_R. Finally, this interval centroid can be defuzzified to obtain the final output value.

B. Exhaustive Defuzzification

The definition of the generalized centroid of the IT2 FS presented in (7), describes the centroid as a composition of individual centroids of all of its embedded fuzzy sets. The exhaustive defuzzification method constructs this interval centroid by enumerating all the embedded fuzzy sets and calculating their respective centroids. This exhaustive algorithm for defuzzification of general T2 FSs can be adapted for the case of IT2 FSs as follows [20]:

Step 1: Enumerate all possible interval type-2 embedded fuzzy sets. The are $n = \prod_{i=1}^{N} M_i$ embedded fuzzy sets, where N is the resolution of the primary domain and M_i is the resolution of the secondary domain at the i^{th} slice.

Step 2: For each embedded fuzzy set find the minimum secondary grade. This is trivial as all secondary membership grades equal to 1.

Step 3: For each embedded fuzzy set calculate the domain value of the centroid of the type-2 embedded fuzzy set.

Step 4: Pair the computed domain value from **Step 3** with the secondary grade of 1.

Step 5: For each unique domain value, the maximum secondary grade is selected. In case of IT2 FSs, this means that only a single record about a defuzzified embedded FS is kept for each unique domain value.

The set of ordered pairs defines the centroid. The crisp output can then be calculated as the average of all calculated centroid coordinates. The average will be uniformly weighted, as the secondary grade of each defuzzified embedded FS equals to 1. Note that if the primary domain is considered to be continuous as opposed to the discretized one, the centroid becomes a continuous interval T1 FS.

C. Karnik-Mendel Iterative Procedure (KMIP)

The centroid of the IT2 FS is an interval T1 FS. According to Karnik and Mendel, it can be completely described by its left and right end points y_L and y_R. As derived by Karnik and Mendel, these boundary points can be expressed as in [20]:

$$y_L = \frac{\sum_{i=1}^{L} x_i \bar{\mu}_{\tilde{B}}(x_i) + \sum_{i=L+1}^{N} x_i \underline{\mu}_{\tilde{B}}(x_i)}{\sum_{i=1}^{L} \bar{\mu}_{\tilde{B}}(x_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{B}}(x_i)} \quad (8)$$

$$y_R = \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\tilde{B}}(x_i) + \sum_{i=R+1}^{N} x_i \bar{\mu}_{\tilde{B}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{B}}(x_i) + \sum_{i=R+1}^{N} \bar{\mu}_{\tilde{B}}(x_i)} \quad (9)$$

Points L and R are important switching points computed by the KMIP algorithm. Using the boundary values of the type-reduced centroid $C(\tilde{B})$ the final crisp defuzzified value y can be computed as the mean of the centroid interval:

$$y = \frac{(y_L + y_R)}{2} \quad (10)$$

The KMIP algorithm calculates the switching points L and R in (8) and (9) and the boundaries y_L and y_R of the centroid of the respective IT2 FS. The description of the KMIP algorithm given below was adopted from [20]. The algorithm consists of two phases, which independently compute the values of y_L and y_R. The algorithm for computing the left boundary y_L can be described in several steps as follows:

Step 1: Initialize a vector of weights w_i as follows:

$$w_i = \frac{1}{2}\left[\bar{\mu}_{\tilde{B}}(x_i) + \underline{\mu}_{\tilde{B}}(x_i)\right] \qquad i = 1,..N \quad (11)$$

And compute the value of y:

$$y = \frac{\sum_{i=1}^{N} x_i w_i}{\sum_{i=1}^{N} w_i} \quad (12)$$

Step 2: Find switching point k ($1 \le k \le N-1$) such that

$$x_{k+1} \le y \le x_{k+1} \quad (13)$$

Step 3: Set

$$w_i = \begin{cases} \bar{\mu}_{\tilde{B}}(x_i) & i \le k \\ \\ \underline{\mu}_{\tilde{B}}(x_i) & i \ge k+1 \end{cases} \quad (14)$$

And compute the value of y' as:

$$y' = \frac{\sum_{i=1}^{N} x_i w_i}{\sum_{i=1}^{N} w_i} \quad (15)$$

Step 4: If $y' = y$, stop and set $y_L = y$ and $L = k$. Otherwise, go to **Step 5**.
Step 5: Set $y = y'$ and go to **Step 2**.

The procedure for computing the value of y_R is identical to computing y_L except that in **Step 3** different update of weights w_i is used as follows:

$$w_i = \begin{cases} \underline{\mu}_{\tilde{B}}(x_i) & i \le k \\ \\ \bar{\mu}_{\tilde{B}}(x_i) & i \ge k+1 \end{cases} \quad (16)$$

The final output value is assigned to y_R and $R=k$.

In [16], the Enhanced KMIP algorithm was presented. However, the proposed enhancements only improved the convergence of the algorithm. As both approaches produce numerically identical results, only the original KMIP algorithm is considered in the rest of this paper.

D. Sampling Defuzzification

The sampling defuzzification method for T2 FS was presented and analyzed in [17], [19], [24] and [27]. This method follows the steps of the exhaustive defuzzification method. The major difference is that only a subset of randomly sampled embedded fuzzy set is considered during the calculation. According to the description in [24] the sampling defuzzification for IT2 FS can be described in several steps as follows:

Step 1: Select the required number of embedded fuzzy sets to be sampled.

Step 2: Repeat for each sample:

Step 2.1: Select an embedded fuzzy set at random.

Step 2.2: Find the domain value by defuzzifying the sampled embedded fuzzy set.

Step 2.3: Find the minimum secondary grade. In case of IT2 FSs this is trivial as the secondary grade equals to 1.

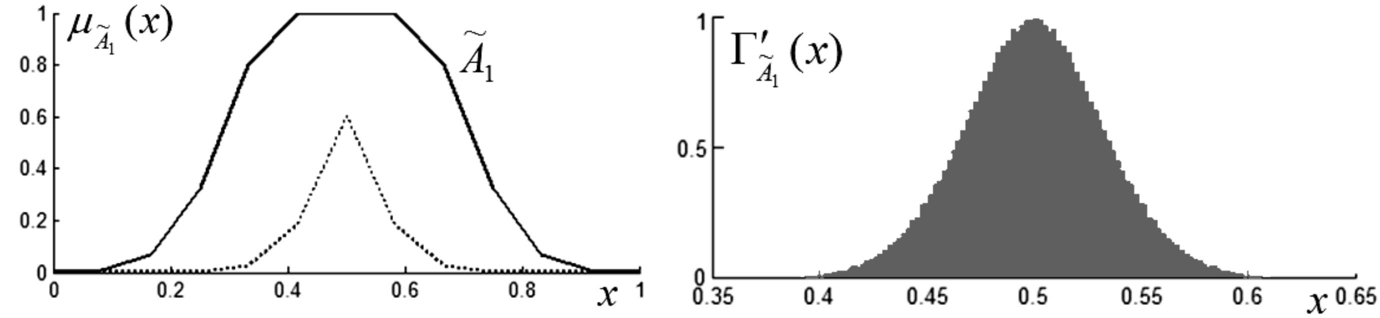
Step 2.4: Add the paired defuzzified value with its secondary grade to the list of pairs constituting the type-reduced centroid.

Step 3: Defuzzify the type-reduced centroid.

The method was experimentally tested, showing a fast convergence towards the expected defuzzification value [17]. The convergence speed was dependent on the cardinality of the sampled set. Extension to the sampling defuzzification method via using the importance sampling technique was proposed in [28]. It was demonstrated that this modification resulted in reduced statistical variance of the computed defuzzified value.

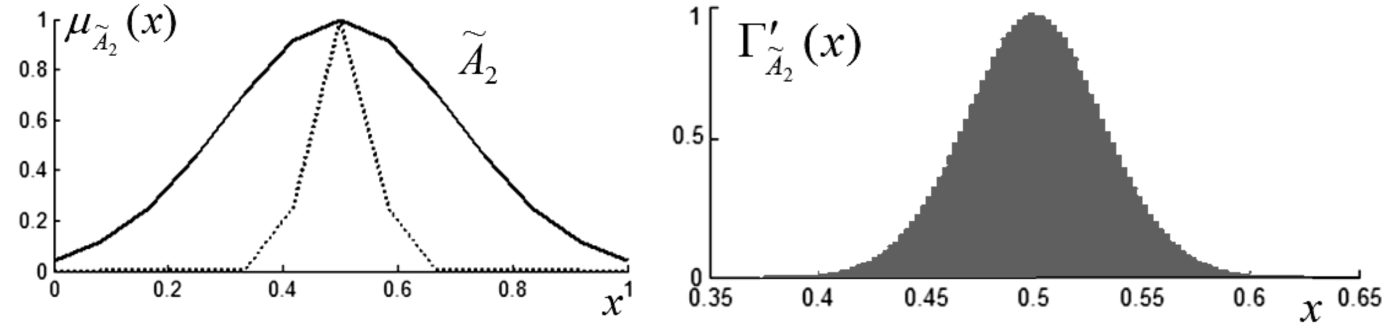
III. Centroid Density Of IT2 Fuzzy Sets

The exhaustive defuzzification method defuzzifies all embedded fuzzy sets and then constructs the interval centroid



(a) (b)

Fig. 1 Interval Type-2 Gaussian fuzzy set with uncertain mean (a) and its corresponding discrete centroid density (b).



(a) (b)

Fig. 2 IT2 Gaussian fuzzy sets with uncertain standard deviation (a) and its corresponding discrete centroid density distribution (b).

as a composition of the calculated domain values. However, in **Step 5** of the exhaustive defuzzification method as described in Section II.B the set of defuzzified embedded fuzzy sets is considerably reduced. This is achieved via application of Zadeh's extension principle, which maps the IT2 FS back to its interval centroid. Here, only a single centroid of the embedded fuzzy sets per each unique domain value remains in the solution and have an impact on the geometrical properties of the final interval centroid. Hence, from the perspective of the type-reduction algorithm, all embedded fuzzy sets, which are defuzzified to an identical domain value form an equivalence class. The members of each such class are here termed *redundant* embedded fuzzy sets.

Definition 1: Two embedded fuzzy sets A_1 and A_2 are redundant if they both defuzzify into an identical domain value as $C(A_1) = C(A_2)$.

As an example, consider a group of embedded fuzzy sets that are vertically shifted in the domain of the secondary variable u. In **Step 5** of the exhaustive defuzzification method, those embedded fuzzy sets are treated as redundant and only a single representative one remains after the type-reduction process.

This redundancy pruning significantly simplifies the complex inner structure of the centroid. This inner structure is similar to the stratified structure of the type-reduced set presented in [19]. This paper further demonstrates that the distribution of redundant embedded fuzzy sets provides an insight into the discrepancies in the convergent behavior of the KMIP algorithm and the sampling defuzzification method [1].

Next, assume that apart from applying the extension principle, also the number of redundant embedded fuzzy sets for each unique domain value is being recorded during the exhaustive defuzzification process. The distribution function $\Gamma_{\tilde{A}}(x)$ of this quantity is denoted here as the centroid density function of the IT2 FS $\tilde{A}$.

Definition 2: The centroid density $\Gamma_{\tilde{A}}(x)$ for domain value x denotes the normalized number of redundant embedded fuzzy sets that are defuzzified to the identical domain value x:

$$\Gamma_{\tilde{A}}(x) = \alpha \sum_{i=1}^{N} f(A_i)\,, \quad f(A_i) = \begin{cases} 0 & C(A_i) \neq x \\ 1 & C(A_i) = x \end{cases} \tag{17}$$

Here, N is the number of all embedded fuzzy sets, A_i is the embedded fuzzy set and α is the normalization factor, which denotes the maximum number of redundant embedded fuzzy sets for any value of the domain variable x, scaling it into the interval between 0 and 1.

Property 1: The value of the centroid density is zero outside the boundaries of the interval centroid:

$$\Gamma_{\tilde{A}}(x) = 0 \quad \forall\, x \notin [y_L, y_R] \tag{18}$$

Proof: By the definition of the KMIP method, bound y_L constitutes the left most location of any centroid of the available embedded fuzzy sets. Similarly, bound y_R is the right most location of any centroid of all available embedded fuzzy sets. Hence, the centroids of all embedded fuzzy sets are

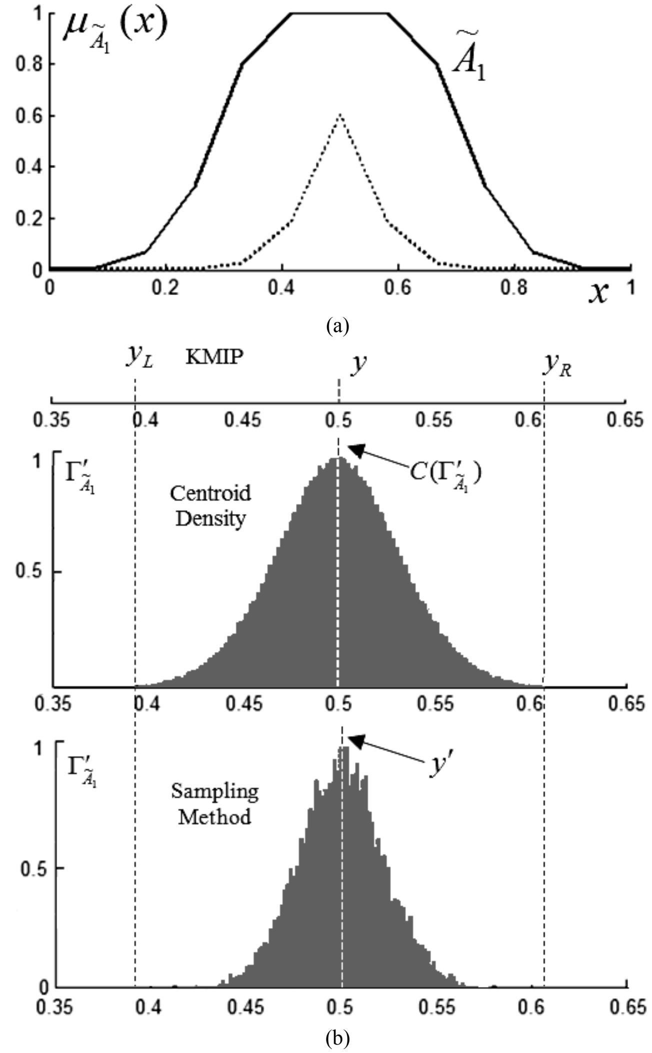


Fig. 3 Symmetric IT2 Gaussian fuzzy set with uncertain mean (a), its centroid density and the defuzzified values (b), y – center of gravity of the centroid density distribution, y' – sampling defuzzifier.

located in this interval and the centroid density is thus zero outside this interval.

From the computational perspective, the discrete centroid density $\Gamma'_{\tilde{A}}(x)$ is more practical, since the primary domain must be discretized for practical implementations.

Definition 3: Assume that the primary domain is discretized into N samples $x_1, x_2, \ldots, x_N$ with an ε distance between two consecutive samples Then the discrete centroid density $\Gamma'_{\tilde{A}}(x_i)$ for value x_i denotes the normalized number of redundant embedded fuzzy sets that are defuzzified into the discretized interval around x_i:

$$\Gamma'_{\tilde{A}}(x_i) = \alpha \sum_{i=1}^{N} f(A_i)\,, \quad f(A_i) = \begin{cases} 0 & C(A_i) \notin [x_i \pm \dfrac{\varepsilon}{2}] \\ 1 & C(A_i) \in [x_i \pm \dfrac{\varepsilon}{2}] \end{cases} \tag{19}$$

Again, the normalization factor α is the maximum number of redundant embedded fuzzy set for any discrete interval in the primary domain, scaling it to the interval between 0 and 1.

Finally, consider the center of gravity of function $\Gamma'_{\tilde{A}}(x)$, which can be denoted as $C(\Gamma'_{\tilde{A}})$ and computed as follows:

$$C(\Gamma'_{\tilde{A}}) = \frac{\sum_{i=1}^{N} \Gamma'_{\tilde{A}}(x_i)\, x_i}{\sum_{i=1}^{N} \Gamma'_{\tilde{A}}(x_i)} \tag{20}$$

An example of centroid density for two different IT2 FSs is presented in Fig. 1 and Fig. 2. The figures show symmetric Gaussian interval type-2 fuzzy set with uncertain mean and uncertain standard deviation, respectively. In both cases, variable x was discretized into 13 samples and each secondary membership function was sampled using 3 samples, leading to a total of 1,594,323 embedded fuzzy sets.

The centroid density function provides valuable insight into the inner structure of the interval centroid. The only available method for computing the centroid density function is the exhaustive defuzzification method that accounts for all available embedded fuzzy sets. However, the applicability of this approach is hindered by the vast amount of embedded fuzzy sets, which makes this method computationally intractable even for modest discretization levels.

The KMIP method offers an opposite approach. This algorithm only considers two special embedded fuzzy sets and only computes the two boundary points y_L and y_R of the interval centroid. In the final step of the KMIP defuzzification approach, the center of gravity of this interval centroid is computed as the mid-point between the boundary values y_L and y_R. One of the greatest advantages of the KMIP approach is its computational speed.

The sampling defuzzifier can be seen as a middle way between the KMIP method and the exhaustive defuzzification approach. This method approximates the final result by considering only a subset of the available embedded fuzzy sets.

IV. Experimental Results

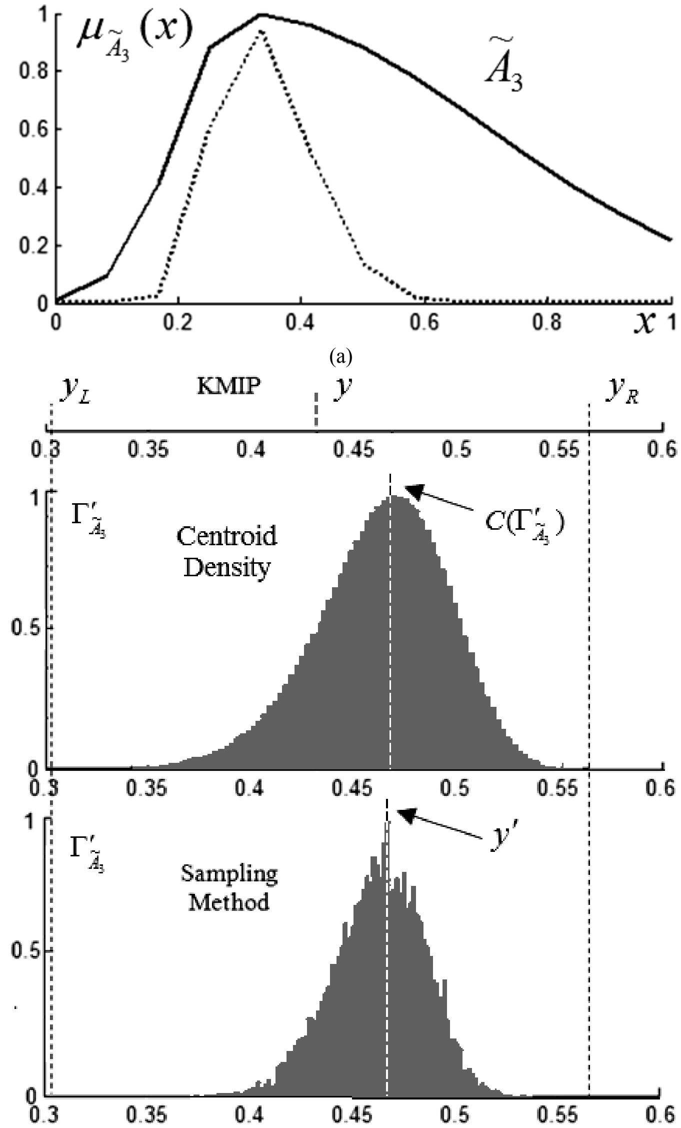
In this section the convergent behavior of the sampling defuzzifier is experimentaly compared to the deterministic result produced with the KMIP. It is shown that unlike the KMIP approach, the sampling algorithm converges towards the center of gravity of the respective centroid density distrubution function. Two types of IT2 FS are considered in the study, symmetric and non-symmetric.

A. Symmetric IT2 fuzzy sets

Consider a symmetric IT2 fuzzy set $\tilde{A}$ the FOU of which is symmetrical about its mean m:

$$\overline{\mu}_{\tilde{A}}(m+x) = \overline{\mu}_{\tilde{A}}(m-x) \tag{21}$$

$$\underline{\mu}_{\tilde{A}}(m+x) = \underline{\mu}_{\tilde{A}}(m-x) \tag{22}$$



(a)

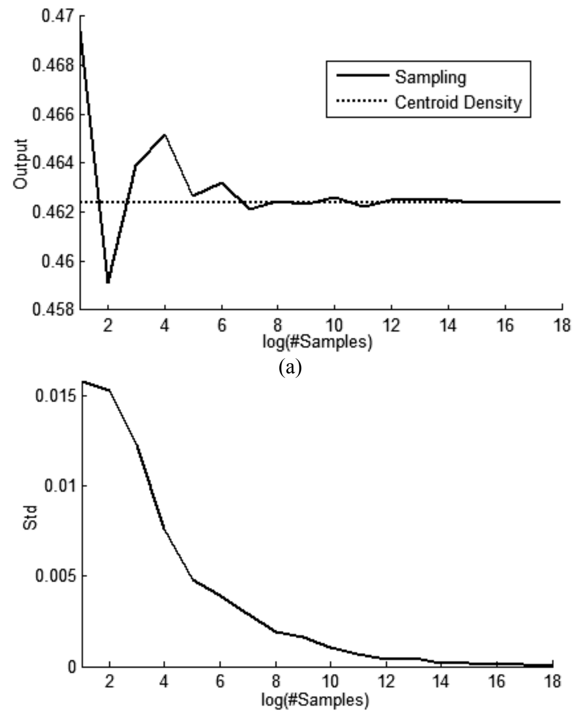
(b)

Fig. 4 Non-symmetric Interval Type-2 Gaussian fuzzy set (a), its centroid density distribution and the defuzzified values (b)), *y* – center of gravity of the centroid density distribution, y*'* – sampling defuzzifier.

TABLE I
COMPARISON OF THE DEFUZZIFIED OUTPUT VALUES

Fuzzy Set	Centroid Density $C(\Gamma'_{\tilde{A}})$	KMIP Method y	KMIP Method y_L	KMIP Method y_R	Sampling Method y'
Symmetric	0.5	0.5	0.3940	0.3001	0.5004 ± 0.0005
Non-Symmetric	0.4624	0.4311	0.6060	0.5621	0.4625 ± 0.0003

A symmetrical Gaussian IT2 FS $\tilde{A}_1$ with uncertain mean is denoted in Fig. 3(a). The domain of FS $\tilde{A}_1$ was again discretized into 13 samples in the *x* domain and 3 samples for each secondary membership function yielding a total of



(a)

(b)

Fig. 5 The convergence of the mean (a) and the standard deviation (b) of the output value produced by the sampling defuzzifier with increasing number of samples.

1,594,323 available embedded fuzzy sets. This FS was first defuzzified using the KMIP algorithm. Next, the centroid density function was computed using the exhaustive defuzzification method evaluating all 1,594,323 embedded fuzzy sets. Finally, the IT2 FS $\tilde{A}_1$ was also defuzzified using the sampling defuzzifier randomly selecting only 10,000 embedded fuzzy sets. The comparison of the defuzzified values and a graphical view of the discrete centroid density function $\Gamma'_{\tilde{A}_1}(x)$ are presented in Table I and Fig. 3(b).

It can be observed, that the centroid density distribution $\Gamma'_{\tilde{A}_1}(x)$ follows approximately a symmetric Gaussian normal distribution. As demonstrated in Table I, the defuzzified values produced by the KMIP algorithm and the center of gravity of the centroid density function are in agreement. The defuzzified value produced by the sampling defuzzifier is shown to be converging towards the correct value. In order to obtain an accurate picture, the average of 20 defuzzification cycles is reported in Table I for the sampling defuzzifier, together with the standard deviation of the result.

B. Non-Symmetric IT2 Fuzzy Sets

Next, a non-symmetric IT2 FS $\tilde{A}_3$ with uncertain standard deviation was considered with its FOU as depicted in Fig. 4(a). All defuzzification methods have been applied with the same parameters as in the previous experiment.

The defuzzified values and the discrete centroid density function $\Gamma'_{\tilde{A}_3}(x)$ are reported in Table I and in Fig. 4(b). It could be observed that despite the centroid density function

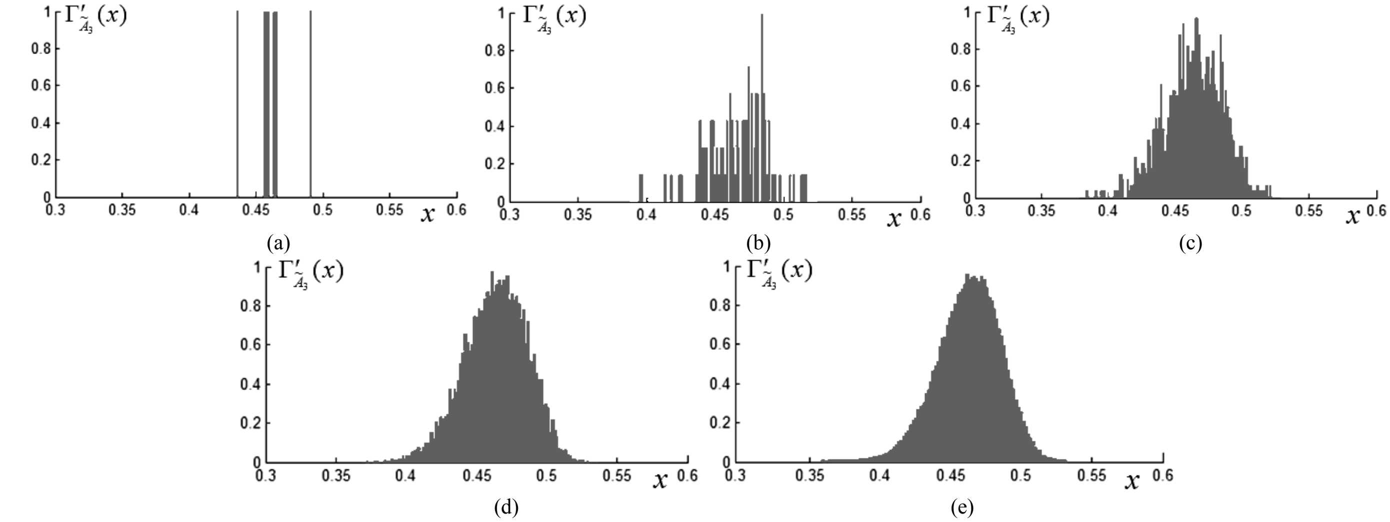


Fig. 6 The sampled centroid density distribution for the sampling defuzzifier using 10, 100, 1.000, 10.000, 100.000 (a)-(e) sampled embedded fuzzy sets.

$\Gamma'_{\tilde{A}_3}(x)$ having the nature of a Gaussian distribution, it is far from being symmetric. This observation can be attributed to the non-symmetric distribution of the width of the FOU of the original IT2 FS. This non-symmetric nature is also reflected in the centroid density function produced by the sampling defuzzification method. Both functions are shifted away from the mid-point of the centroid interval computed by the KMIP algorithm. The actual defuzzified values can be compared in Table I.

As it can be seen from the shown examples, given a non-symmetric IT2 FS, which is a typical case for the output fuzzy sets of IT2 FLSs, a significant difference between the defuzzified values produced by the KMIP algorithm and the sampling defuzzifier can be expected. The interpretation of these discrepancies is an open issue.

Recently, novel representations for general T2 fuzzy sets using α -planes [7] and *zSlices* [5] have been introduced. In [24] the results of defuzzification of α -plane based general T2 FS using the KMIP algorithm, sampling defuzzification and exhaustive defuzzification have been reported. It was demonstrated that for increasing number of α -planes the KMIP and the sampling method did not converge to identical values. The authors believe that the introduced concept of centroid density can provide further insight and explanations into such discrepancies.

C. Convergence of the Sampling Defuzzification

An additional set of experiments have been carried out in order to verify the convergence of the sampling defuzzifier towards the center-of-gravity of the centroid density function, which was computed using the exhaustive defuzzification technique. The sampling defuzzifier was applied to the non-symmetric IT2 FS $\widetilde{A}_3$ depicted in Fig. 4(a). The number of sampled embedded fuzzy sets ranged from 2^2 to 2^{18}. Each experiment was repeated 20 times. The mean and the standard deviation of the defuzzified values are reported in Fig. 5. In addition, Fig. 6(a)-(e) visually demonstrates the convergence of the sampled centroid density function. From the presented results it can be noted that the sampling defuzzification method provides a steady convergence towards the center of gravity of the centroid density function.

Hence, it can be concluded that the sampling defuzzifier converges to the output value obtained by applying the weighted average to the centroid density function. On the other hand, the KMIP method computes the mid-point of the interval centroid calculated with accordance to the extension principle. The interval centroid does not maintain information about the centroid density function, since it was removed during the application of Zadeh's extension principle.

V. Conclusion

This paper presented a comparative analysis of the performance of two fundamentally different defuzzification techniques for IT2 FSs. The stochastic sampling defuzzification method and the deterministic KMIP method were considered in this work. The concepts of redundant embedded fuzzy sets and the notion of centroid density function of an IT2 FS were introduced. These novel concepts provided explanation for some of the discrepancies between the results produced by the KMIP method and the sampling defuzzifier.

It was shown that the sampling defuzzification approach converges towards the center of gravity of the centroid density function, which can be computed using the exhaustive defuzzification method. On the other hand, the KMIP algorithm calculates the results as the mid-point of the interval centroid computed with accordance to the extension principle. Since the application of the extension principle during the type-reduction removes the information about the centroid density, the KMIP and the sampling defuzzification techniques inevitable converge towards different results. This difference becomes especially significant for non-symmetric IT2 FS.

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