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Importance Sampling Based Defuzzification for General Type-2 Fuzzy Sets

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Abstract—General type-2 fuzzy logic systems (T2 FLS) constitute a powerful tool for coping with ubiquitous uncertainty in many engineering applications. However, the immense computational complexity associated with defuzzification of general T2 fuzzy sets still remains an unresolved issue and prohibits its practical use. This paper proposes a novel importance sampling based defuzzification method for general T2 FLS. Here, a subset from the domain of all embedded fuzzy sets is randomly sampled using a specific probability distribution function. The algorithm is compared with the previously published uniform sampling defuzzification method. Experimental results demonstrate that importance sampling substantially reduces the variance of the sampling defuzzification method. Comparison of T2FLS output surfaces showed that smoother and more stable response can be achieved with the proposed importance sampling based defuzzification method.

I. Introduction

TYPE-2 fuzzy logic systems (T2 FLS) constitute a powerful technique for dealing with the ubiquitous uncertainty in a variety of engineering applications [1]. By their nature, Type-1 (T1) FLS can cope with the linguistic uncertainty originating in the imprecise meaning of words. In addition, T2 FLS can also manage the uncertainty about input measurements and the input data used to tune the constructed systems [2].

A schematic view of T2 FLS is depicted in Fig. 1 [2]. The system is identical to a T1 FLS with two exceptions: i) T2 fuzzy sets (T2 FS) are used in either antecedents or consequents of the implemented fuzzy rules, and ii) the T1 defuzzification stage is substituted with the output processing block. The output T2 FS needs to be first reduced into a T1 FS. The reduced T1 FS is then defuzzified into a crisp output value. This paper addresses the output processing stage of the fuzzy inference engine.

The computational efforts associated with the inference process and especially with the type-reduction phase are prohibitively large [3]. Hence, general T2 FS are rarely used in engineering application, despite of their theoretical potentials [4]-[6]. Several solutions have been proposed. Simplification of general T2 FS into Interval T2 FS by Karnik and Mendel enabled the use of T2 FS in praxis [3], [7]. Further, Coupland and John exploited the computational geometry to speed-up the inferencing and the defuzzification process of general T2 FS [8], [9]. The same authors also proposed a sampling defuzzifier, which provided an approximate solution [10]. Recently, Liu offered a computationally effective defuzzification strategy using the α-plane representation of general T2 FS [11], [12].

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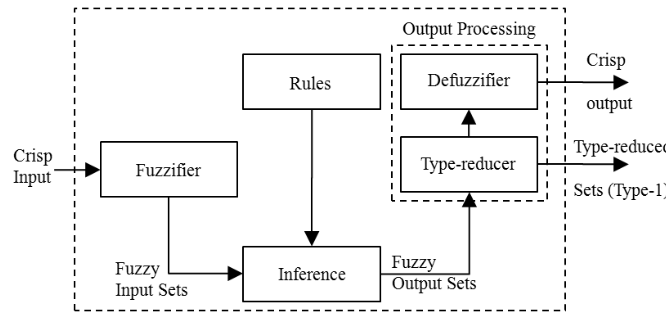

Fig. 1 Type-2 fuzzy logic system [1].

This paper proposes a novel importance sampling based defuzzification algorithm for general T2 FS. It is inspired by the previously published sampling defuzzification method [10]. The importance of different embedded sets is modeled using a Gaussian probability distribution function. The parameters of the sampling function are computed based on the FOU of the general T2 FS. When compared to the original sampling defuzzification method, the implemented importance sampling strategy reduces variance of the calculated values and gives a smoother response of the respective T2 FLS.

The rest of the paper is organized as follows. Section II provides a background overview of T2 FLS. Section III describes the proposed importance sampling based defuzzification method. Experimental results are presented in Section IV. The paper is concluded in Section V.

II. General Type-2 Fuzzy Sets

This section reviews fundamentals about general type-2 fuzzy sets. A general type-2 fuzzy set (T2 FS) $\widetilde{A}$ can be expressed using its type-2 membership function $\mu_{\widetilde{A}}(x,u)$, where $x \in X$ and $u \in J_x$ [2]:

$$\widetilde{A} = \int_{x\in X}\int_{u\in J_x} \mu_{\widetilde{A}}(x,u)/(x,u) \quad J_x \subseteq [0,1] \tag{1}$$

Here, operator $\iint$ denotes union over all possible values of x and u, and $0 \le \mu_{\tilde{A}}(x,u) \le 1$. Two representations of T2 FS are commonly adopted; the *vertical-slice* representation and the *wavy-slice* representations.

First, consider the *vertical-slice* representation. By using a specific value for $x = x'$ a vertical slice $\mu_{\tilde{A}}(x',u)$ of function $\mu_{\tilde{A}}(x,u)$ can be obtained. This vertical slice defines a secondary membership function $\mu_{\tilde{A}}(x = x',u)$ for $x' \in X$ and $\forall u \in J_{x'} \subseteq [0,1]$:

$$\mu_{\tilde{A}}(x = x',u) \equiv \mu_{\tilde{A}}(x') = \int_{u \in J_{x'}} f_{x'}(u)/u \quad J_{x'} \subseteq [0,1] \quad (2)$$

Here, $f_{x'}(u)$ denotes the secondary grade or amplitude of the secondary membership function and $0 \le f_{x'}(u) \le 1$. Assuming that the X-domain is discretized using N samples the T2 FS $\tilde{A}$ can be reformulated as a composition of its all vertical slices:

$$\tilde{A} = \sum_{i=1}^{N} \left[\int_{u \in J_{x_i}} f_{x_i}(u)/u \right] \Big/ x_i \quad (3)$$

Next, consider the *wavy-slice* representation. The T2 FS $\tilde{A}$ is a composition of embedded T2 FS $\tilde{A}_e$. For a discrete universe of discourse having N elements, $\tilde{A}_e$ can be described as:

$$\tilde{A}_e = \sum_{i=1}^{N} \left[f_{x_i}(\theta_i)/\theta_i \right] \big/ x_i \quad \theta_i \in J_{x_i} \subseteq U \in [0,1] \quad (4)$$

Using the *wavy-slice* representation the T2 FS $\tilde{A}$ can be described as a union of all its n embedded T2 FSs:

$$\tilde{A} = \bigcup_{j=1}^{n} \tilde{A}_e^j \quad (5)$$

For the discretized X-domain, the centroid $C_{\tilde{A}}$ of T2 FS $\tilde{A}$ can be calculated using the *Extension Principle* and by enumerating all embedded fuzzy sets [2], [3]:

$$C_{\tilde{A}} = \int_{\theta_1 \in J_{x_1}} \ldots \int_{\theta_N \in J_{x_N}} \left[f_{x_1}(\theta_1) \wedge \ldots \wedge f_{x_N}(\theta_N) \right] \Bigg/ \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i} \quad (6)$$

Here, every possible combination of variables $\theta_1,\ldots,\theta_N$ form an embedded T2 FS, which has a secondary grade of $f_{x_1}(\theta_1) \wedge \ldots \wedge f_{x_N}(\theta_N)$. Operator $\wedge$ is the specific t-norm used. The minimum t-norm is considered in this work.

Assuming that each domain J_{x_i} was discretized into M_i points, the number of embedded sets that have to be enumerated in (6) is $n = \prod_{i=1}^{N} M_i$. Already for small number of discretization steps, n reaches inadmissibly large values.

The crisp output value y can be obtained by applying one of the available defuzzification methods to the type-reduced fuzzy sets $C_{\tilde{A}}$. Here, the centroid defuzzifier is used [2]:

$$y = \frac{\sum_{i=1}^{n} x_i C_{\tilde{A}}(x_i)}{\sum_{i=1}^{n} C_{\tilde{A}}(x_i)} \quad (7)$$

III. Importance Sampling Based Defuzzification

This section first reviews the previously published uniform sampling defuzzification method for general T2 FS. Next, the importance sampling based defuzzification algorithm is introduced.

A. Sampling Method for Type-2 Defuzzification

A sampling method for type-2 defuzzification was presented in [10]. A subset of embedded sets with specified cardinality was randomly chosen from all possible embedded sets. The centroid $C_{\tilde{A}}$ introduced in (6) was then computed using only those sampled embedded sets. Next, a centroid defuzzifier was applied to obtain a crisp output y based on the type-reduced type-1 fuzzy set $C_{\tilde{A}}$ (7).

The method was experimentally tested, showing a fast convergence towards the expected defuzzification value [10]. However, as it was later demonstrated in work of the same authors, the variance of the sampling process had a negative impact on the quality of the final control surface [13].

B. Importance Sampling Based Defuzzification

Importance sampling is a well established technique for variance reduction widely used in Monte Carlo methods [14]. The rationale behind importance sampling is that different values of the used random variable have various effects on the estimated system. By emphasizing the more important values the variance of the sampling process can be reduced. This is achieved via choosing a suitable Probability Distribution Function (PDF) for the random variable. A suitable PDF will redistribute the values of the random variable so that their sampling frequency matches their importance in the domain.

From (7) it can be observed that the importance of each sampled embedded fuzzy set $\tilde{A}_e$ with respect to the centroid defuzzifier is given by its secondary grade in $C_{\tilde{A}}(x_i)$. This secondary grade of particular embedded fuzzy sets $\tilde{A}_e$ is calculated as the t-norm $f_{x_1}(\theta_1) \wedge \ldots \wedge f_{x_N}(\theta_N)$ in (6). Using the minimum t-norm leads to the following observation:

- The importance of each sampled embedded fuzzy set $\tilde{A}_e$ with respect to the centroid defuzzifier (7) is determined by its minimum secondary grade $f_{x_i}(\theta_i)$ (6).

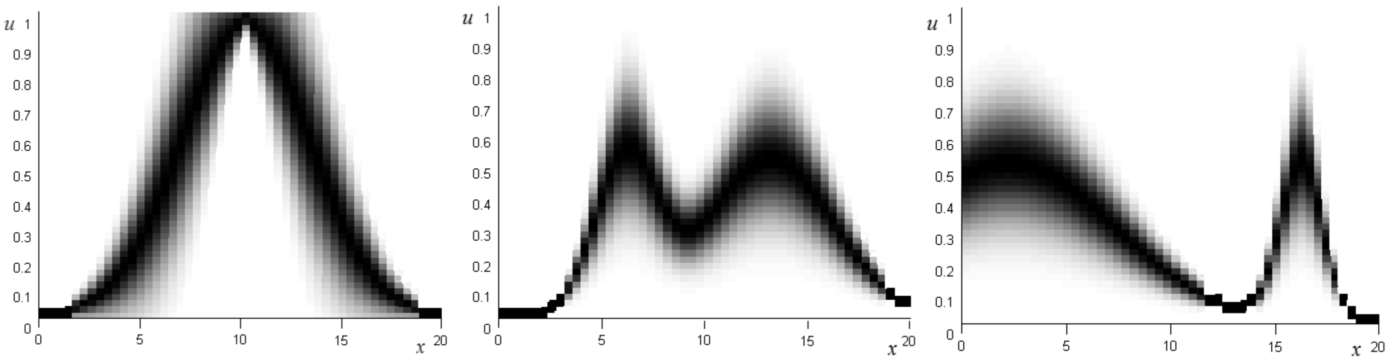

Fig. 2 Type-2 fuzzy sets used for defuzzification testing. Both axis x and u are discretized using 50 points.

Using this observation, the presented importance based sampling defuzzification method uses a Gaussian PDF for sampling embedded fuzzy sets in each domain J_{x_i}. The parameters of the PDF are adjusted according to the secondary membership function $\mu_{\tilde{A}}(x)$. The mean $m_{pdf}(x)$ and the standard deviation $\sigma_{pdf}(x)$ are calculated based on the footprint of uncertainty (FOU). By projecting the T2 FS $\tilde{A}$ into the x-u plane a bounded region of secondary uncertainty is obtained. The FOU can be expressed as the union of all secondary domains [2]:

$$FOU(\tilde{A}) = \bigcup_{x \in X} J_x \qquad (8)$$

The upper and lower membership functions $\overline{\mu_{\tilde{A}}}(x)$ and $\underline{\mu_{\tilde{A}}}(x)$ are the bounds for the FOU of $\tilde{A}$. They can be expressed as:

$$\overline{\mu_{\tilde{A}}}(x) \equiv \overline{FOU(\tilde{A})} \qquad \forall x \in X \qquad (9)$$

$$\underline{\mu_{\tilde{A}}}(x) \equiv \underline{FOU(\tilde{A})} \qquad \forall x \in X \qquad (10)$$

However, when a continuous unbounded secondary membership functions $\mu_{\tilde{A}}(x)$ (e.g. Gaussian) are used, the FOU covers the whole domain x-u. To reduce the sampling region, the concept of α -*plane* is applied [11], [12]. The α -*plane* $\tilde{A}_\alpha$ of T2 FS $\tilde{A}$ is the union of all primary memberships whose secondary grades are greater than or equal to a specific value of α $(0 \le \alpha \le 1)$ [11], [12]:

$$\tilde{A}_\alpha = \int_{\forall x \in X} \int_{\forall u \in J_x} \left\{ (x,u) \middle| \, f_x(u) \ge \alpha \right\} \qquad (11)$$

Hence, by choosing a specific value of α the FOU at the α -*plane* for unbounded secondary membership functions can be redefined as:

$$FOU_\alpha(\tilde{A}) = FOU(\tilde{A}_\alpha) \qquad (12)$$

For the sake of clarity, the general FOU (8) is a specific α -*plane* for $\alpha = 0$. In a similar manner, the upper and lower bounds for FOU_α can be formulated as:

$$\overline{\mu^{\alpha}_{\tilde{A}}}(x) \equiv \overline{FOU_\alpha(\tilde{A})} = \overline{\tilde{A}_\alpha} \qquad \forall x \in X \qquad (13)$$

$$\underline{\mu^{\alpha}_{\tilde{A}}}(x) \equiv \underline{FOU_\alpha(\tilde{A})} = \underline{\tilde{A}_\alpha} \qquad \forall x \in X \qquad (14)$$

The lower and upper boundaries are used for computing the parameters of the sampling PDF. The mean $m_{pdf}(x)$ of the Gaussian PDF can be calculated as the mean of the lower and the upper bounds of FOU_α:

$$m_{pdf}(x) = \frac{\overline{\mu^{\alpha}_{\tilde{A}}}(x) + \underline{\mu^{\alpha}_{\tilde{A}}}(x)}{2} \qquad (15)$$

The standard deviation $\sigma_{pdf}(x)$ of the Gaussian PDF is obtained as a modulated width of the margin between $\overline{\mu^{\alpha}_{\tilde{A}}}(x)$ and $\underline{\mu^{\alpha}_{\tilde{A}}}(x)$:

$$\sigma_{pdf}(x) = \frac{\left| \overline{\mu^{\alpha}_{\tilde{A}}}(x) - \underline{\mu^{\alpha}_{\tilde{A}}}(x) \right|}{2^\beta} \qquad (16)$$

Parameter β controls the spread of the Gaussian PDF. It effectively adjusts the amount of secondary uncertainty to be considered in the defuzzification process. Two boundary cases can be considered. First, when β approaches positive infinity:

$$\lim_{\beta \to \infty} \sigma_{pdf}(x) = 0 \qquad (17)$$

In this case, the Gaussian PDF reduces to a unit probability of selecting the mean value $m_{pdf}(x)$. The type reduction collapses to a simple selection of the principal membership function.

Second, when the value of β approaches negatuve infinity:

$$\lim_{\beta \to -\infty} \sigma_{pdf}(x) = \infty \qquad (18)$$

TABLE I
COMPARISON OF UNIFORM AND IMPORTANCE SAMPLING BASED DEFUZZIFICATION FOR T2 FS FROM FIG. 2(A).

Number of Samples	Uniform Sampling		Importance Sampling $\beta=2$		Importance Sampling $\beta=4$		Importance Sampling $\beta=8$		Time [s]
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	
10	10.0041	0.0774	10.0082	0.0594	10.0018	0.0326	10.0025	0.0265	0.0174
50	9.9959	0.0352	9.9850	0.0303	9.9989	0.0165	10.0009	0.0116	0.0187
100	9.9934	0.0230	10.0008	0.0212	10.0015	0.0111	9.9999	0.0077	0.0213
200	10.0021	0.0168	10.0008	0.0161	9.9990	0.0074	9.9992	0.0060	0.0310
500	10.0001	0.0130	10.0000	0.0103	9.9990	0.0048	10.0005	0.0036	0.0564
1,000	10.0003	0.0067	9.9997	0.0060	9.9997	0.0034	10.0000	0.0026	0.111
10,000	9.9994	0.0022	10.0000	0.0024	10.0001	0.0011	10.0000	0.0082	0.9792

TABLE II
COMPARISON OF UNIFORM AND IMPORTANCE SAMPLING BASED DEFUZZIFICATION FOR T2 FS FROM FIG. 2(B).

Number of Samples	Uniform Sampling		Importance Sampling $\beta=2$		Importance Sampling $\beta=4$		Importance Sampling $\beta=8$		Time [s]
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	
10	10.5401	0.0654	10.5382	0.0457	10.5324	0.0228	10.5551	0.0196	0.0174
50	10.5561	0.0289	10.5402	0.0197	10.5354	0.0110	10.5526	0.0085	0.0189
100	10.5596	0.0194	10.5452	0.0150	10.5359	0.0078	10.5539	0.0060	0.0295
200	10.5494	0.0127	10.5454	0.0102	10.5350	0.0061	10.5528	0.0042	0.0461
500	10.5492	0.0077	10.5450	0.0058	10.5361	0.0039	10.5535	0.0032	0.0892
1,000	10.5494	0.0058	10.5453	0.0041	10.5354	0.0028	10.5528	0.0019	0.1521
10,000	10.598	0.0017	10.5454	0.0015	10.5357	0.0009	10.5529	0.0006	1.5031

TABLE III
COMPARISON OF UNIFORM AND IMPORTANCE SAMPLING BASED DEFUZZIFICATION FOR T2 FS FROM FIG. 2(C).

Number of Samples	Uniform Sampling		Importance Sampling $\beta=2$		Importance Sampling $\beta=4$		Importance Sampling $\beta=8$		Time [s]
	Mean	Std	Mean	Std	Mean	Std	Mean	Std	
10	7.5075	0.0908	7.4590	0.0737	7.4664	0.0388	7.4835	0.0241	0.0158
50	7.5078	0.0377	7.4684	0.0312	7.4684	0.0163	7.4860	0.0122	0.0232
100	7.5048	0.0333	7.4667	0.0221	7.4682	0.0122	7.4851	0.0065	0.0286
200	7.5024	0.0181	7.4711	0.0154	7.4665	0.0086	7.4866	0.0055	0.0585
500	7.5061	0.0124	7.4690	0.0100	7.4669	0.0061	7.4861	0.0040	0.0923
1,000	7.5062	0.0075	7.4695	0.0074	7.4674	0.0038	7.4860	0.0028	0.1737
10,000	7.5055	0.0028	7.4697	0.0024	7.4670	0.0013	7.4861	0.0008	1.5041

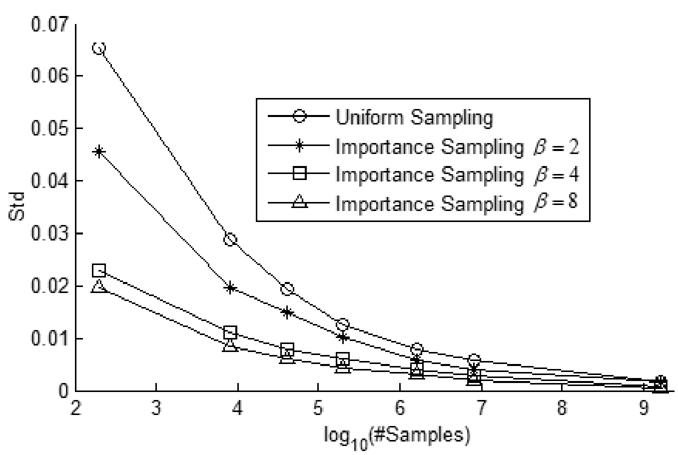

Fig. 3 Standard deviation as a function of the number of samples for uniform sampling and for the proposed importance based sampling defuzzifcation method.

In this case, the Gaussian PDF reduces to a uniform PDF and the proposed method becomes equivalent to the previously published uniform sampling defuzzification [10]. The effects of different values of β are demonstrated in the following section.

It is important to note, that the proposed importance sampling based defuzzification is not limited by the considered Gaussian PDF used of sampling. The Gaussian PDF was chosen here, because it resembles the typical distribution of secondary membership grades along each vertical slice. Generalization of the proposed method for an arbitrary sampling PDF is the scope of future work.

IV. EXPERIMENTAL TESTING

This section presents experimental comparison of the importance sampling and the uniform sampling defuzzifier.

A. Defuzzification Performance

For the purpose of experimental comparison three T2 FS were constructed. They are depicted in Fig. 2. All three fuzzy sets were constructed using Gaussian primary and secondary membership functions. The grey tone displays the secondary membership grade, the darker the color, the higher the grade. The domain was discretized using 50 points along both x and u dimensions. Fig. 2(a) shows a symmetric Gaussian T2 FS, for which the defuzzification value is known as 10 (mean of the primary membership function). The defuzzification values for T2 FSs in Fig. 2(b) and Fig. 2(c) are not known.

The experimental testing compared the performance of the previously proposed uniform sampling defuzzifier and the new

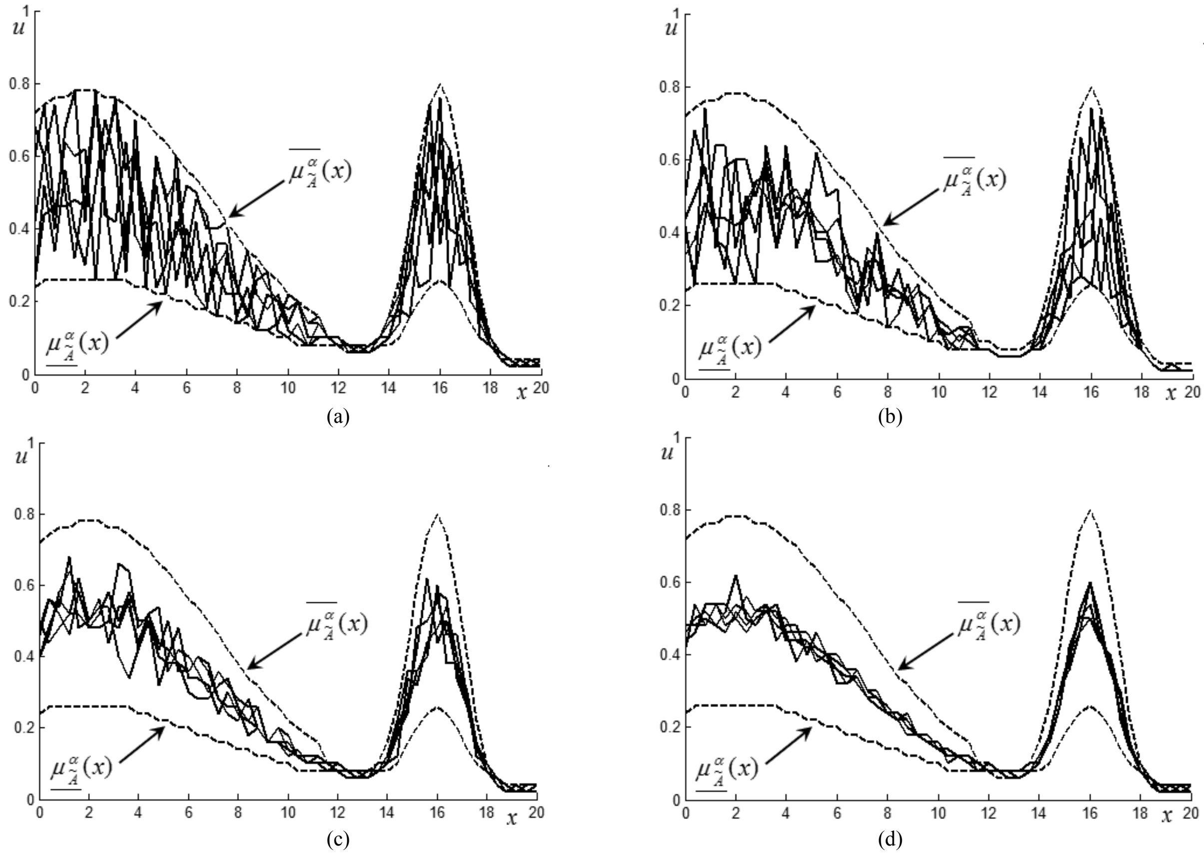

Fig. 4 Five sampled embedded fuzzy sets using uniform sampling (a) and using importance sampling for $\beta = 2, 4, 8$ (b)-(d).

importance sampling defuzzification method. Both mean and standard deviation of the output values over 100 runs of the defuzzification procedure were analyzed. Each defuzzifier was tested for different number of samples: 10, 50, 100, 200, 500, 1,000 and 10,000. The footprints of uncertainty FOU_α of all sets were constructed at $\alpha = 0.01$. This value is also used in all the following experiments. Three importance sampling defuzzifiers were tested with $\beta = 2, 4, 8$.

Table I, II and III show the obtained results for respective fuzzy sets in Fig. 2(a)-2(c). It can be observed that all importance sampling defuzzifiers converged to nearly identical output values. It can be further seen, that the standard deviation of the computed results is lower for the proposed importance sampling based defuzzification and it decreases with increasing β. For better visualization of the computed results, the standard deviation from Table II has been plotted as a function of the number of samples in Fig. 3. Further, the FOU_α and five sampled embedded sets for the T2 FS from Fig. 2(c) are depicted in Fig. 4(a)-(d). It can be observed that different values of β influence the selection of particular embedded sets.

In addition, the defuzzification time is displayed in Table I, II and III. Because the importance sampling does not increase the time complexity of the computation, only the averaged defuzzification time for all the methods is displayed. It should be pointed out that the computation time does not scale linearly for smaller number of samples (<200). This is because for lower number of samples the computational time is governed by computing the upper and lower boundaries of FOU. For the sake of completeness, the algorithms were implemented in MATLAB and run on Intel Core 2 Duo with 2.00 GHz and 2.00 GB or RAM.

B. Output Surface Analysis

The standard deviations of the computed results listed in Table I, II and III might seem negligible when compared to the actual defuzzified value. However, this variance can have a significant impact on the stability (smoothness) of the final control surface of the T2 FLS. To visualize such effects the output surfaces of simple T2 FLSs were compared [12].

A simple T2 FLS was constructed. It consists of two Mamdani type rules:

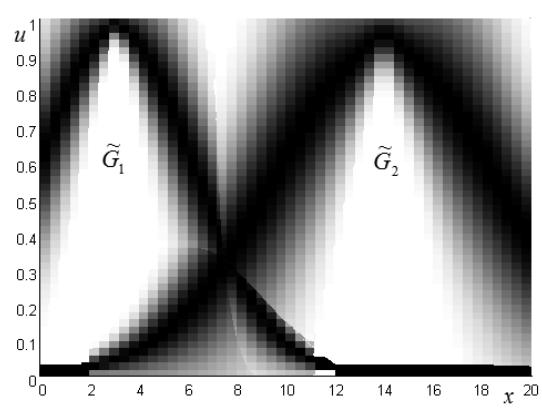

Fig. 5 Type-2 rule consequents $\tilde{G}_1$ and $\tilde{G}_2$.

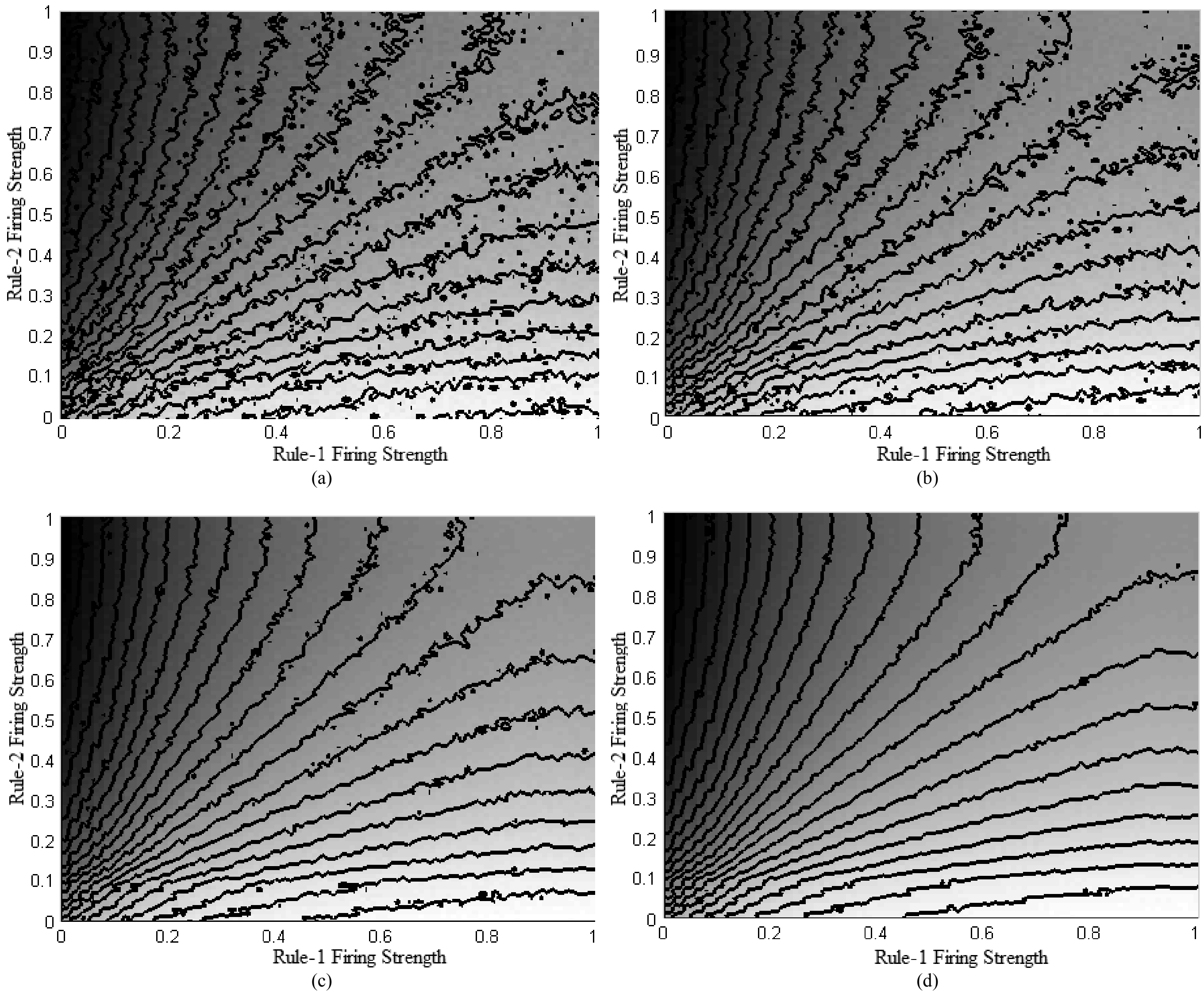

Fig. 6 Comparison of the output surfaces for the sampling defuzzifier (a) and the importance sampling based defuzzifier with $\beta = 2, 4, 8$ (b)-(d).

- **Rule 1**: *IF* x_1 *is* $\widetilde{F}_1$ *THEN y IS* $\widetilde{G}_1$
- **Rule 2**: *IF* x_2 *is* $\widetilde{F}_2$ *THEN y IS* $\widetilde{G}_2$

Here, x_1 and x_2 are the input variables, $\widetilde{F}_1$ and $\widetilde{F}_2$ are antecedent fuzzy T2 sets, and $\widetilde{G}_1$ and $\widetilde{G}_2$ are consequent fuzzy T2 sets. The rules consequents $\widetilde{G}_1$ and $\widetilde{G}_2$ are displayed in Fig. 5. The outputs of both rules are combined using the join operation [2]. The output surface of this T2 FLS is given by all possible combinations of Rule 1 and Rule 2 firing strengths. Each rule can fire with strength between 0 and 1. Because the rule firing strengths are of type-2 they are modeled as a Gaussian functions centered at the selected type-1 firing degree with $\sigma = 0.05$. By discretizing each domain into 100 samples the output surface can be obtained. Since only the different firing strengths of Rule 1 and Rule 2 are considered, the actual parameters of rule antecedents $\widetilde{F}_1$ and $\widetilde{F}_2$ do no matter.

Output surfaces for the uniform defuzzifier and the importance sampling defuzzifier with $\beta = 2, 4, 8$ are displayed in Fig. 6(a)-(d). All defuzzifiers sampled 10 embedded sets from the aggregated output of the T2 FLS. Again, $\alpha = 0.01$ was used for constructing the FOU_α. The grey tone of particular point denotes the defuzzified value. In order to better assess the characteristics (smoothness) of the output surfaces, 20 isolines are plotted on the surfaces.

By analyzing the nature of the isolines it can be concluded that the output surface provided by the uniform sampling defuzzifier (Fig. 6(a)) features rather instable and noisy behavior. This is caused by the high variance of the computed defuzzified values. In comparisons, the output surface for the importance sampling based defuzzifier with $\beta = 4$ and $\beta = 8$ features substantially reduced variance.

V. Conclusion

This paper presented a novel importance sampling based defuzzification method for general T2 FS. The proposed algorithm is inspired by the previously published uniform sampling defuzzifier. Gaussian PDF is used as a mapping function for the importance sampling process. Parameters of the Gaussian PDF are computed based on the FOU of the T2 FS.

The presented results showed that the importance sampling reduces the variance in the defuzzified results. It was illustrated that substantially smoother and more stable output surface of a T2 FLS can be obtained when the importance sampling defuzzification technique is used. Generalization of the proposed method for an arbitrary sampling PDF is a subject of future work.

REFERENCES

[1] L. A. Zadeh, "The Concept of a Linguistic Variable and its Approximate Reasoning - II," in *Information Sciences*, No. 8, pp. 301-357, 1975.

[2] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, Upper Saddle River, NJ, 2001.

[3] N. N. Karnik, J. M. Mendel, "Type-2 Fuzzy Logic Systems," in *IEEE Trans. on Fuzzy Systems*, vol. 7, no. 6, pp. 643-658, December 1999.

[4] H. A. Hagras, "A Hierarchical Type-2 Fuzzy Logic Control Architecture for Autonomous Mobile Robots," in *IEEE Trans. Fuzzy Systems*, vol. 12, no. 4, pp. 524-539, 2004.

[5] J. Figueroa, J. Posada, J. Soriano, M. Melgarejo, S. Rojas, "A type-2 fuzzy logic controller for tracking mobile objects in the context of robotic soccer games," in *Proc. IEEE Intl' Conf. on Fuzzy Systems*, pp. 359-364, 2005.

[6] Ch. Lynch, H. Hagras, "Using Uncertainty Bounds in the Design of an Embedded Real-Time Type-2 Neuro-Fuzzy Speed Controller for Marine Diesel Engines," in *Proc. IEEE Intl' Conf. on Fuzzy Systems*, pp. 1446-1453, Vancouver, Canada, 2006.

[7] J. M. Mendel, R. John, F. Liu, "Interval Type-2 Fuzzy Logic Systems Made Simple," in *IEEE Trans. on Fuzzy Systems*, vol. 14, no. 6, pp. 808-821, 2006.

[8] S. Coupland, R. John, "A Fast Geometric Method for Defuzzification of Type-2 Fuzzy Sets," in *IEEE Trans. on Fuzzy Systems*, vol. 16, no. 4, pp. 929-941, 2008.

[9] S. Coupland, R. John, "Geometric Type-1 and Type-2 Fuzzy Logic Systems," in *IEEE Trans. on Fuzzy Systems*, vol. 15, no. 1, pp. 3-15, February 2007.

[10] S. Greenfield, R John, S. Coupland, "A Novel Sampling Method for Type-2 Defuzzification," in *Proc. UKCI 06*, pp. 120-127, 2005.

[11] F. Liu, "An efficient centroid type-reduction strategy for general type-2 fuzzy logic system," in *Information Sciences*, vol. 178, pp 2224-2236, 2008.

[12] J. M. Mendel, F. Liu, D. Zhai, "α -Plane Representation for Type-2 Fuzzy Sets: Theory and Applications," in IEEE Trans. on Fuzzy Systems, vol. 17, no. 5, pp. 1189-1207, October 2009.

[13] S. Coupland, R. John, "An Investigation into Alternative Methods for the Defuzzification of an Interval Type-2 Fuzzy Set," in *Proc. IEEE Intl' Conf. on Fuzzy Systems*, pp. 1425-1432, Vancouver, Canada, 2006.

[14] R. Srinivasan, *Importance sampling – Applications in communications and detection*, Springer-Verlag, Berlin, 2002.