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A note: all homogeneous second order difference equations of degree one have semiconjugate factorizations

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We show that the second order difference equation $x_{n+1} = f_n(x_n, x_{n-1})$ on a group G has an i-semiconjugate factorization into a triangular system of first order equations if and only if each mapping f_n is homogeneous of degree 1.

Keywords: Semiconjugate; Homogeneous; Non-autonomous; Difference equation

Let G be a given nontrivial group and consider the non-autonomous second order difference equation

$$x_{n+1} = f_n(x_n, x_{n-1}), \quad n = 0, 1, 2, \ldots \tag{1}$$

where $x_n \in G$ and $f_n : G \times G \to G$ for each n. This equation recursively generates a *solution* or *orbit* $\{x_n\}_{n=-1}^{\infty}$ in G.

The group structure provides a suitable framework for our main result (see the Remarks below). However, in most applications of this result G is a substructure of a more complex object such as a vector space or an algebra possessing a compatible or natural metric topology relative to which the mappings f_n may be continuous. In such cases, each f_n may be defined on the ambient structure as long as the following invariance condition holds

$$f_n(G \times G) \subset G \quad \text{for all } n. \tag{2}$$

A function $f : G \times G \to G$ is *homogeneous of degree one* or *HD1* for short, if

$$f(tx, ty) = tf(x, y) \quad \text{for all } t, x, y \in G. \tag{3}$$

If G is not commutative, then equation (3) defines a "left version" of HD1 with the right version analogously defined. For later reference, we also define f to be *HD0* or *homogeneous of degree zero* if $f(tx, ty) = f(x, y)$ for all $t, x, y \in G$. We note in passing that homogeneous functions of any real degree may be defined in the context of the real number system as solutions of *functional equations* similar to equation (3); see Ref. [1] for more details on this issue.

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Some examples of HD1 functions with G being the set $(0, \infty)$ of all positive real numbers under ordinary multiplication are the following:

$$|ax + by|, \sqrt{ax^2 + bxy + cy^2}, \frac{ax^2 + bxy + cy^2}{\alpha x + \beta y}.$$

In the first example above $a, b \in \mathbb{R}$ (not both zero) and in the last two $a, b, c, \alpha, \beta \geq 0$ with $a + b + c, \alpha + \beta > 0$. These functions are defined on larger subsets of the plane $\mathbb{R}^2$ than $(0, \infty)^2$ but they satisfy equation (2) with the multiplicative group $G = (0, \infty)$. There are large classes of HD1 and HD0 functions on groups as the following easy-to-check result shows.

PROPOSITION. *Let G be a nontrivial group.*

(a) *If* $g : G \to G$ *is any given function, then* $g(y^{-1}x)$ *is a HD0 function on* $G \times G$ *and each of* $xg(y^{-1}x)$ *and* $yg(y^{-1}x)$ *is HD1.*
(b) *The composition* $f(g_1(x, y), g_2(x, y))$ *is HD1 if* $f, g_1, g_2 : G \times G \to G$ *are HD1.*
(c) If f *is HD1 and* g *is HD0, then* $fg(x, y) = f(x, y)g(x, y)$ *is HD1.*

A function $f : G \times G \to G$ is semiconjugate to $h : G \to G$ if there is a link function $\lambda : G \times G \to G$ such that

$$\lambda(f(x, y), x) = h(\lambda(x, y)). \tag{4}$$

The function h is often called a semiconjugate factor of f. We refer to Ref. [6] for general facts about semiconjugacy as well as some examples. In this note we are interested in the special link function $\lambda(x, y) = y^{-1}x$. With this link function, we say that f is *i-semiconjugate* (or *i-sc* for short) to h (i is for group inversion). Equation (4) takes the form

$$x^{-1}f(x, y) = h(y^{-1}x). \tag{5}$$

LEMMA. *Let G be a nontrivial group. Then* $f : G \times G \to G$ *is i-sc to a mapping* $h : G \to G$ *if and only if f is HD1.*

Proof. If f is HD1 then for each $u \in G$ define $h(u) = f(1, u^{-1})$ where 1 is the group identity. Then for each $x, y \in G$ we have $h(y^{-1}x) = f(x^{-1}x, x^{-1}y) = x^{-1}f(x, y)$ and i-semiconjugacy follows. Conversely, if f is *i-sc* to a mapping h as specified, then for every $t \in G$,

$$f(tx, ty) = txh((ty)^{-1}tx) = txh(y^{-1}x) = tf(x, y).$$

so that f is HD1. □

Remarks.

1. The above Lemma characterizes i-semiconjugacy in terms of the HD1 property in the group setting. On the other hand, the definition of HD1 functions in equation (3) above does not require the group inversion, so it is valid in the broader context of *semigroups*. But currently a semiconjugate factorization result analogous to the above Lemma for HD1 functions on semigroups is not known.

2. Consider the semigroup $\mathbb{R}$ of all real numbers under ordinary multiplication. The linear function $\ell(x, y) = ax + by$ (a, b nonzero constants) is clearly HD1 with respect to this semigroup, but since 0 is non-invertible ℓ cannot have an i-sc factorization on $\mathbb{R}$. Also, ℓ does not have an i-sc factorization on the multiplicative group $\mathbb{R} - \{0\}$ because ℓ does not satisfy (2). On the other hand, ℓ does have an i-sc factorization on the multiplicative group $(0, \infty)$ if $a, b > 0$; furthermore, semiconjugate factorizations (not related to the HD1 property) are generally known for linear transformations on $\mathbb{R}^n$ (including the unfolding of ℓ); see Ref. [6].

THEOREM. *Let G be a nontrivial group. If all* f_n *in equation* (1) *are HD*1 *then equation* (1) *is equivalent to the following system of first order equations*

$$u_{n+1} = h_n(u_n) \tag{6a}$$

$$v_{n+1} = v_n u_{n+1} \tag{6b}$$

where each function h_n *is the i-sc factor of* f_n *for every n.*

Proof. For each solution $\{x_n\}_{n=-1}^{\infty}$ of equation (1) define $u_n = x_{n-1}^{-1} x_n$ for each $n = 0, 1, 2, \ldots$ Then $x_{n+1} = x_n u_{n+1}$ and

$$u_{n+1} = x_n^{-1} x_{n+1} = x_n^{-1} f_n(x_n, x_{n-1}) = f_n(1, x_n^{-1} x_{n-1}) = h_n(u_n).$$

It follows that $\{u_n\}_{n=0}^{\infty}$ is a solution of equation (6a) so that $\{(u_n, v_n)\}_{n=0}^{\infty}$ is a solution of (6a,b) with $v_n = x_n$ for $n = 0, 1, 2, \ldots$

Conversely, let $\{(u_n, v_n)\}_{n=0}^{\infty}$ be a solution of equation (6a,b). Then $\{u_n\}_{n=0}^{\infty}$ is a solution of equation (6a). Choose $x_{-1} \in G$ and set $x_n = v_n$ for $n = 0, 1, 2, \ldots$ Then $x_{n+1} = v_{n+1} = v_n u_{n+1} = x_n h_n(u_n)$ so that

$$x_{n+1} = x_n f_n(1, u_n^{-1}) = f_n(x_n, x_n[x_{n-1}^{-1} x_n]^{-1}) = f_n(x_n, x_{n-1}).$$

It follows that the sequence $\{x_n\}_{n=-1}^{\infty}$ is a solution of equation (1).

Note that the system of equation (6a,b) is "triangular" since equation (6a) does not depend on the second variable v_n. For general results on the periodic solutions of such systems, see Ref. [2]. Equation (6b) gives

$$v_n = v_0 u_1 u_2 \ldots u_n \quad n = 1, 2, 3, \ldots \tag{7}$$

in terms of a given solution $\{u_n\}_{n=0}^{\infty}$ of the first order equation (6a).

For HD1 functions, the above theorem essentially reduces the study of the second order equation (1) to that of the first order equation (6a). The following examples illustrate this idea in simple settings. For a more detailed example involving the absolute value function on the real line we refer to Ref. [3]. □

Example 1. Let G be the group of positive real numbers $(0, \infty)$ under ordinary multiplication and the usual metric. Consider the non-autonomous rational difference equation

$$x_{n+1} = \frac{a_n x_n^2 + b_n x_n x_{n-1}}{c_n x_{n-1}} \tag{8}$$

where $a_n, b_n \geq 0$ with $a_n + b_n > 0$ and $c_n > 0$ for all n. Define $\alpha_n = a_n/c_n$ and $\beta_n = b_n/c_n$ and note that $f_n(x, y) = (\alpha_n x^2 + \beta_n xy)/y$ is HD1 for all n. By the above theorem each f_n is i-semiconjugate on the group G to $h_n(u) = f_n(1, 1/u) = \alpha_n u + \beta_n$. With the linear maps h_n equation (6a) can be solved explicitly. Then with the aid of equation (7) an explicit solution for equation (8) can be obtained if desired.

Example 2. This example illustrates the additive case. Let G be the group of all real numbers under ordinary addition and the usual metric. Consider the second order difference equation

$$x_{n+1} = a_n + x_n + c_n(b_n + x_n - x_{n-1})^2 \tag{9}$$

where a_n, b_n and c_n are given sequences of real numbers. Here the functions $f_n(x, y) = a_n + x + c_n(b_n + x - y)^2$ are HD1 on the additive group G since for each real number t,

$$f_n(t + x, t + y) = a_n + t + x + c_n(b_n + x - y)^2 = t + f_n(x, y).$$

The i-sc factors are $h_n(u) = f_n(0, -u) = a_n + c_n(b_n - u)^2$ and equation (7) gives $x_n = x_0 + \sum_{k=1}^{n} u_n$. To illustrate the possibility of complex behavior for equation (9) with minimum calculations, consider the special case where $a_n = a$, $b_n = 0$ and $c_n = -1$ for all n with $1 \leq a \leq 2$. The quadratic map $u_{n+1} = a - u_n^2$ has an invariant interval $[-a, a]$ and exhibits deterministic chaos when a is sufficiently near the value 2 due to the emergence of a snap-back repeller [5] at the unique interior fixed point in $[-a, a]$ (or with a close enough to 2, the appearance of a period 3 solution [4]). Each solution $\{u_n\}_{n=0}^{\infty}$ defines a complicated orbit $\{x_n\}_{n=-1}^{\infty}$ which if unfolded in the phase plane will be confined within the strip $y - a < x < y + a$.

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