Pebbling Graphs

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1 Introduction.

Consider an arbitrary digraph G, with pebbles placed on some of its vertices. Suppose that, for any directed edge (v, w) of G, we are allowed to change the configuration of pebbles by removing two pebbles from v and placing one pebble on w. Then for a vertex v of G, if n exists such that, however n pebbles are placed on G, one pebble can always be moved to v, we let $f(v, G)$ be the smallest such n. The *pebbling number* of an undirected graph G, $f(G)$, is $\max_{v \in V(G)} f(v, G)$. It is conjectured by Ronald Graham (see [1]) that for all graphs G and H, $f(G \times H) \leq f(G)f(H)$. We will prove this when G and H are trees, and compute $f(G)$ exactly for some graphs G.

2 Pebbling products.

Let T be a tree and let v be a vertex of T. Let T_v^* be the rooted tree obtained from T by directing all edges towards v. A *path-partition* of a rooted tree U is a partition of the edges of U such that each set of edges in the partition forms a directed path. A *maximum path-partition* of a rooted tree U with height n is a path-partition P of U such that every path in P touches a leaf, and, for all $0 \leq m \leq n$, if we consider the vertex-induced subtree U' of U induced by the set of all leaves of level m or greater and ancestors of these leaves, then $\{P_0 \in P | P_0 \subseteq E(U')\}$ is a path-partition of U'. The *path-size sequence* of a path-partition $\{P_1, \ldots, P_n\}$ is an n-tuple $(a_1, \ldots, a_n)$, where a_j is the number of edges in P_j and the P_j's are ordered so that $a_1 \geq a_2 \geq \ldots \geq a_n$. If we have a maximum path-partition P of a rooted tree U with height h, let U' be the subtree of U induced by all leaves of level h and their ancestors. Then some subset of P is a path-partition of U', so some path in P must run from a leaf of level h to the root. Hence if $(a_1, \ldots, a_n)$ is the path-size sequence of a maximum path-partition of U, we must have $a_1 = h$.

If we have a digraph G with some pebbles placed on it, we let p be the total number of pebbles on G and q be the number of vertices of G with an odd number of pebbles. If G is a digraph and v is a vertex of G, we say that (G, v) can be (α, β, γ)-pebbled if

1. For all $m \geq 1$, if $p \geq \alpha + (m-1)\beta$, then m pebbles can be moved to v.
2. For all $m \geq 2$, if $p + q > 2\gamma + (m-2)\beta$, then m pebbles can be moved to v.

Theorem 1 *Let U be a rooted tree with root v and let G be a digraph with w a vertex of G. If (G, w) can be (α, β, γ)-pebbled, then $(U \times G, (v, w))$ can be $(X + \alpha, \beta 2^h, X + \max(\alpha, \gamma))$-pebbled, where*

$$X = \gamma \sum_{j=1}^{p} d_j + \beta \sum_{j=1}^{p} (2^{d_j} - d_j - 1),$$

h is the height of U, and $(d_1, \ldots, d_p)$ is the path-size sequence of any maximum path-partition of U.

Proof. We induct on h. If h is zero, the result follows trivially. Otherwise let P be a maximum path-partition of U, and let U' be the subtree of U induced by the set of all vertices of level less than h. Then $\{P_0 \cap E(U') \neq \emptyset | P_0 \in P\} = P'$, say, is a path-partition of U'. It can easily be seen that P' is maximum. Let $v_1, \ldots, v_n$ be the vertices in U which are parents of leaves of level h. Then in P' there is a path to each v_j, P_j say; let a_j be the number of edges in P_j. If $b_1, \ldots, b_m$ are the lengths of the remaining paths in P', then the induction hypothesis says that $((v, w), U' \times G)$ can be $(X' + \alpha, \beta 2^{h-1}, X' + \max(\alpha, \gamma))$-pebbled, where

$$X' = \gamma(\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j) + \beta \sum_{i=1}^{n} (2^{a_i} - a_i - 1) + \beta \sum_{j=1}^{m} (2^{b_j} - b_j - 1).$$

Now U can be obtained from U' by adding leaves to $v_1, \ldots, v_n$. Suppose we add L leaves in all. Then the path-partition P must consist of the paths of lengths $b_1, \ldots, b_m$, combined with the paths $P_1, \ldots, P_n$, each augmented by an edge from v_j to one of its leaves, and $L-n$ one-edge paths from the v_j's to their other leaves. Hence we have

$$\begin{aligned} X &= \gamma(\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j + L) + \beta \sum_{i=1}^{n}(2^{a_i+1} - a_i - 2) + \beta \sum_{j=1}^{m}(2^{b_j} - b_j - 1) \\ &= X' + \gamma L + \beta \sum_{i=1}^{n}(2^{a_i} - 1). \end{aligned}$$

Now since P is maximum, the augmented P_j's together with these $L-n$ one-edge paths must form a path-partition for the subtree U_0 of U induced by the leaves of degree h and their ancestors. Hence if we let U_0' be the subtree of U' induced by the v_j's and their ancestors, the P_j's must form a path-partition of U_0'. Then if we set G equal to the trivial graph and use the induction hypothesis, since the trivial graph can be $(1,1,1)$-pebbled, we find that U_0' can be $(Q, 2^{h-1}, Q)$-pebbled, where

$$Q = \sum_{i=1}^{n} 2^{a_i} - n + 1.$$

Now, first, let $m \geq 1$. We will prove that if $p \geq X + \alpha + \beta 2^h(m-1)$, then m pebbles can be moved to (v, w). Let $l_1, \ldots, l_L$ be the level-h leaves of U, and let p' be the number of pebbles in $U' \times G$ and p_k be the number of pebbles in $\{l_k\} \times G$, where $1 \leq k \leq L$. Define q' and q_k similarly. Then if

$$p' + \sum_{k=1}^{L} \frac{p_k - q_k}{2} \geq X' + \alpha + \beta 2^{h-1}(m-1),$$

we will be done, since for each vertex $x \in L_k$, $y \in G$, we can take two pebbles off (x, y) and put one pebble on some (v_i, y). Hence for each k, we can take $p_k - q_k$ pebbles from $\{l_k\} \times G$ and move $(p_k - q_k)/2$ pebbles into $U' \times G$. Then we will have a total of at least $X' + \alpha + \beta 2^{h-1}(m-1)$ pebbles in $U' \times G$, and will be able to move one pebble to (v, w), by the induction hypothesis.

Otherwise, since $p \geq X + \alpha + \beta 2^h(m-1)$, we will have

$$\sum_{k=1}^{L} \frac{p_k + q_k}{2} \quad = \quad p - p' - \sum_{k=1}^{L} \frac{p_k - q_k}{2}$$

$$
\begin{aligned}
&> (X+\alpha+\beta 2^h(m-1)) - (X'+\alpha+\beta 2^{h-1}(m-1)) \\
&= (X-X') + \beta 2^{h-1}(m-1),
\end{aligned}
$$

so

$$\sum_{k=1}^{L}(p_k+q_k) > 2(X-X') + \beta 2^h(m-1).$$

Now for each k, if $p_k+q_k > 2\gamma+(r-2)\beta$, we can take pebbles from $\{l_k\}\times G$ and move r pebbles to (l_k, w), by hypothesis. Hence if $(p_k+q_k-2\gamma)/2\beta > r \geq 0$, we can move $2r+2$ pebbles to (l_k, w) and then $r+1$ pebbles from (l_k, w) into $\{v_1, \ldots, v_n\} \times \{w\}$, so we can move at least $(p_k+q_k-2\gamma)/2\beta$ pebbles into $U_0' \times \{w\}$. But then, doing this for all k, we can put at least

$$\sum_{k=1}^{L} \frac{p_k+q_k-2\gamma}{2\beta}$$

pebbles in $U_0' \times \{w\}$, and then

$$
\begin{aligned}
\sum_{k=1}^{L} \frac{p_k+q_k-2\gamma}{2\beta} &= \frac{\sum_{k=1}^{L}(p_k+q_k) - 2\gamma L}{2\beta} \\
&> \frac{2(X-X') + \beta 2^h(m-1) - 2\gamma L}{2\beta} \\
&= \frac{2\beta\sum_{i=1}^{n}(2^{a_i}-1) + \beta 2^h(m-1)}{2\beta} \\
&= \sum_{i=1}^{n}(2^{a_i}-1) + 2^{h-1}(m-1) \\
&= Q - 1 + 2^{h-1}(m-1),
\end{aligned}
$$

so we can move m pebbles to (v, w), since U_0' can be $(Q, 2^{h-1}, Q)$-pebbled.

Now we will prove that for all $m \geq 2$, if $p+q > 2\max(\alpha,\gamma) + 2X + (m-2)\beta 2^h$, then m pebbles can be moved to (v, w). If $p'+q' > 2\max(\alpha,\gamma) + 2X' + (m-2)\beta 2^{h-1}$, then we are done, by the induction hypothesis. Otherwise, suppose that for some $l \in \{3, \ldots, m\}$,

$$
\begin{aligned}
2\max(\alpha,\gamma) + 2X' + (l-2)\beta 2^{h-1} - q' &\geq p' \\
&> 2\max(\alpha,\gamma) + 2X' + (l-3)\beta 2^{h-1} - q'.
\end{aligned}
$$

Then we can move $l-1$ pebbles to (v,w) in $U'\times G$. Also, we have

$$\begin{aligned} p+q-p'-q' &> 2(X-X')+(m-2)\beta 2^h-(l-2)\beta 2^{h-1} \\ &= 2(X-X')+(2m-l-2)\beta 2^{h-1}, \end{aligned}$$

so as in the first part, we can move $\lfloor (2m-l-2)/2\rfloor+1$ pebbles to (v,w), giving a total of

$$\left\lfloor \frac{2m-l-2}{2} \right\rfloor + 1 + l - 1 = m-1+l-\left\lceil \frac{l}{2} \right\rceil,$$

but since $l\geq 2$, $1+l/2\leq l$ and we are done. If

$$2\max(\alpha,\gamma)+2X'-q'\geq p'\geq X'+\alpha,$$

then we can proceed as above with $l=2$.

Suppose finally that $p'<X'+\alpha$. Then we claim that we can move $X'+\alpha-p'$ pebbles into $U'\times G$ and still leave enough pebbles in $(U\setminus U')\times G$ to move $m-1$ pebbles to (v,w). To move $X'+\alpha-p'$ pebbles into $U'\times G$ at all, we need to have

$$\sum_{k=1}^{L}\frac{p_k-q_k}{2}\geq X'+\alpha-p'. \tag{1}$$

After moving c pebbles out of the $\{l_k\}\times G$'s, the q_k's will still be the same as before, because we remove pebbles by 2's, but the sum of the p_k's will decrease by $2c$. Hence to have enough pebbles left over, we need

$$\sum_{k=1}^{L}(p_k+q_k)-2(X'+\alpha-p')>2(X-X')+\beta 2^h(m-2). \tag{2}$$

For (1) to hold, we need

$$\sum_{k=1}^{L}p_k-\sum_{k=1}^{L}q_k+2p'\geq 2X'+2\alpha, \tag{3}$$

but

$$\begin{aligned} \sum_{k=1}^{L}p_k-\sum_{k=1}^{L}q_k+2p' &\geq p'+q'+\sum_{k=1}^{L}p_k-\sum_{k=1}^{L}q_k \\ &= p+q-2\sum_{k=1}^{L}q_k \\ &\geq p+q-2L|V(G)|. \end{aligned}$$

Now if we put one pebble on each vertex of G, no movement can be done, so we cannot move 2 pebbles to any vertex. In this case $p+q=2|V(G)|$. Hence $\gamma \geq |V(G)|$, and

$$\begin{aligned}\sum_{k=1}^{L} p_k - \sum_{k=1}^{L} q_k + 2p' &\geq p+q-2L\gamma \\ &> 2\max(\alpha,\gamma)+2X+\beta 2^h(m-2)-2L\gamma.\end{aligned}$$

But $X - X' \geq \gamma L$, so (3) holds.

For (2), we need

$$\sum_{k=1}^{L}(p_k+q_k)+2p' > 2X+2\alpha+\beta 2^h(m-2),$$

but this follows from

$$\sum_{k=1}^{L}(p_k+q_k)+2p' \geq \sum_{k=1}^{L}(p_k+q_k)+p'+q' = p+q,$$

so we are done. ∎

Theorem 2 *If we have a graph G with a certain configuration of pebbles and a vertex v of G and wish to move m pebbles to v, then there always exists an acyclic orientation H for G such that m pebbles can still be moved to v in H. Furthermore, if G is a tree, we can take $H = G_v^*$; hence for all trees T and vertices v of T, $f(v,T) = f(v,T_v^*)$.*

Proof. Suppose we have a graph G with pebbles on its vertices, and suppose we wish to move m pebbles to v. Take a sequence of directed edges, $(e_1,\ldots,e_p)$, where pebbling along each e_j in sequence moves m pebbles to v. Now suppose we allow negative numbers of pebbles to reside on vertices, so that pebbling along (x,y) is always possible, and subtracts 2 from the pebble count on x and adds 1 to the pebble count on y. Then if, in $(e_1,\ldots,e_p)$, we find an edge (x,y) followed by (y,x), or a cycle of directed edges, delete the pair or cycle. After pebbling along $(e_1,\ldots,e_p)$ the counts are all nonnegative and at least m pebbles end up on v. Deleting pairs or cycles only increases these counts, since a pair or cycle has the net effect of removing 1 pebble from each of its vertices. After we have deleted all pairs and cycles present, then, we are left with a sequence of edges $(f_1, \ldots, f_n)$ that puts m pebbles on v when pebbled along, except that a pebble count on some vertex may be temporarily negative. But since there are no cycles present, if we order the vertices of G by $v \prec w$ if there is an edge f_j from v to w, and then take the transitive closure, we obtain a partial order. Extend this to a linear order $\prec$ of $V(G)$. Then if we reorder the f_j's such that (x, y) is placed before (z, w) if $x \prec z$, no edge (y, z) can occur before an edge (x, y), since if it did we would have $y \prec x$ and $x \prec y$. Hence in this reordering of the f_j's, a vertex y is only pebbled out of after all pebbling into it has been done; then there are no problems with intermediate counts being negative, and if we pick our orientation H to have $E(H) = \{(x, y) | x \prec y, \{x, y\} \in E(G)\}$, we will be able to move m pebbles to v in H.

If G is a tree, we can always choose to direct all edges towards v, for if not, let (x, y) be an edge directed away from v. Then no pebbles can ever pass from the subtree of G rooted at y into the rest of G (which contains v), so any pebbling steps along (x, y) can be omitted without decreasing the number of pebbles arriving at v. Then if (x, y) is never pebbled, we might as well direct it the other way. Repeating this, we can direct all edges in G towards v. ∎

If we have a digraph G with pebbles on its vertices, we denote the number of pebbles on a vertex v of G by $\mathcal{N}(v)$.

Theorem 3 *Let G be a digraph and let $S \subseteq V(G)$. If $v \in S$ and*

$$\sum_{d(w,v)<\infty,\ w \in S} \mathcal{N}(w) 2^{-d(w,v)} < m, \tag{4}$$

then m pebbles cannot be moved to v by pebbling within S.

Proof. The left-hand side of (4) cannot increase when a pebbling move is made within S, since when we move from vertex w to vertex w', $d(w', v) \geq d(w, v) - 1$. But if m pebbles were on v, then the left-hand side of (4) would be at least m, so we must not be able to move m pebbles to v. ∎

Theorem 4 *Let U be a rooted tree and let v be the root of U. If the path-size sequence of a maximum path-partition for U is $(a_1, \ldots, a_n)$, then*

$$f(v, U) = \sum_{i=1}^{n} 2^{a_i} - n + 1.$$

Furthermore, U can be

$$\left(\sum_{i=1}^{n} 2^{a_i} - n + 1, 2^{a_1}, 2^{a_1} + \sum_{i=2}^{n} 2^{a_i - 1}\right) - \textit{pebbled.} \tag{5}$$

Proof. Let $\{P_1, \ldots, P_n\}$ be a maximum path-partition for U. The a_j edges in P_j will touch $a_j + 1$ vertices. Let $Q_j \subseteq V(U)$ contain the a_j of these vertices furthest from v, and let v_j be the vertex in Q_j furthest from v. The Q_j's are disjoint and do not contain v. Put $2^{a_j} - 1$ pebbles on v_j for each j. With this initial configuration of pebbles, we cannot move 2 pebbles from Q_j to the vertex of Q_j nearest the root (by Theorem 3), so we cannot move pebbles out of any Q_j, and in particular, we cannot put a pebble on v, which is not in any Q_j. Hence

$$f(v, U) > \sum_{i=1}^{n} 2^{a_i} - n.$$

Now in Theorem 1, set G equal to the trivial graph. Then G can be $(1, 1, 1)$-pebbled, so (U, v) can be

$$\left(\sum_{i=1}^{n} 2^{a_i} - n + 1, 2^{a_1}, \sum_{i=1}^{n} 2^{a_i} - n + 1\right) - \text{pebbled.} \tag{6}$$

In particular, $f(v, U)$ is as desired. Now suppose we have put some pebbles on U. Since for every $w \in V(U)$ there is only one pebbling move that can be made out of w, the number of pebbles we will be able to move to v will not be increased if we take a pebble of some vertex w and put 2 pebbles on one of w's children (since we will just move them back to w), and it clearly will not be decreased. Now if $q > n$, there are at least $q - n$ nonleaf vertices containing at least one pebble; we can take one pebble off each of these vertices and put 2 pebbles on one of the children of each of these vertices without affecting the number of pebbles we can move to v. But doing this increases p by $q - n$, and by (6), if

$$p \geq \sum_{i=1}^{n} 2^{a_i} - n + 1 + 2^{a_1}(m - 1),$$

we can move m pebbles to v. Hence if

$$p + q \geq \sum_{i=1}^{n} 2^{a_i} - n + 1 + 2^{a_1}(m - 1) + n,$$

we can also move m pebbles to v, so we can take

$$2\gamma = \sum_{i=1}^{n} 2^{a_i} - n + 1 + 2^{a_1} + n - 1, \text{ or } \gamma = 2^{a_1} + \sum_{i=2}^{n} 2^{a_i - 1},$$

as desired. ∎

Theorem 5 *If $T_1, \ldots, T_n$ are (undirected) trees, then*

$$f(T_1 \times \ldots \times T_n) \leq f(T_1) \ldots f(T_n).$$

Proof. We will show that for all $v_j \in V(T_j)$,

$$f((v_1, \ldots, v_n), T^*_{1v_1} \times \ldots \times T^*_{nv_n}) \leq f(v_1, T^*_{1v_1}) \ldots f(v_n, T^*_{nv_n}). \quad (7)$$

This will imply the desired result, since by Theorem 2, $f(v, T^*_v) = f(v, T)$, if T is a tree and v is a vertex of T. Theorem 1 tells us that for a digraph G and a vertex w of G, if (G, w) can be (α, α, α)-pebbled, then for a rooted tree U with root v, $(U \times G, (v, w))$ can be $(X, 2^{b_1}\alpha, X)$-pebbled, where

$$X = \alpha(\sum_{j=1}^{m} 2^{b_j} - m + 1),$$

and $(b_1, \ldots, b_m)$ is the path-size sequence of a maximum path-partition for U. Then $X \geq 2^{b_1}\alpha$, so $(U \times G, (v, w))$ can be (X, X, X)-pebbled, and by Theorem 4, $X = \alpha f(v, U)$. Then, since the trivial graph can be $(1, 1, 1)$-pebbled, we can induce to find that for rooted trees $U_1, \ldots, U_n$ with roots $v_1, \ldots, v_n$, $((v_1, \ldots, v_n), U_1 \times \ldots \times U_n)$ can be (Y, Y, Y)-pebbled, where

$$Y = f(v_1, U_1) \ldots f(v_n, U_n).$$

This implies (7), so we are done. ∎

3 Exact pebbling numbers.

Let P^*_n be the digraph with $V(P^*_n) = \{p_1, \ldots, p_n\}$ and $E(P^*_n) = \{(p_1, p_2), \ldots, (p_{n-1}, p_n)\}$.

Theorem 6 *Let U be a rooted tree and let v be the root of U. If the path-size sequence of a maximum path-partition for U is $(a_1, \ldots, a_n)$, then*

$$f((v, p_m), U \times P^*_m) = 2^{m-1+a_1} + (m-1)\sum_{j=2}^{n} 2^{a_j - 1} + \sum_{j=2}^{n} 2^{a_j} - (n-1). \quad (8)$$

Proof. Setting $G = U$ and $U = P_n^*$ in Theorem 1, and using (5), we find that the left-hand side of (8) is no bigger than the right-hand side. For the other direction, let Q_j and v_j be as in Theorem 4. Put $2^{a_j} - 1$ pebbles on (v_j, p_1) and 2^{a_j-1} pebbles on (v_j, p_k), where $j = 2, \ldots, n$ and $k = 2, \ldots, m$. Also, put $2^{m-1+a_1} - 1$ pebbles on (v_1, p_1). Then we claim that no pebble can be moved out of $Q_j \times P_m^*$, $j = 2, \ldots, n$, and that in $(Q_1 \cup \{v\}) \times P_m^*$, no pebble can be moved to (v, p_m). This will imply that in (8), the left-hand side is at least as big as the right-hand side. Suppose a pebble could be moved out of $Q_j \times P_m^*$ for some $j > 1$. Then we would have to move two pebbles to some (w_j, p_k), where w_j is the vertex of Q_j nearest to v and $k \in \{1, \ldots, m\}$. But

$$\begin{aligned}
&\sum_{x \in Q_j \times P_m^*,\ d(x,(w_j,p_k))<\infty} \mathcal{N}(x) 2^{-d(x,(w_j,p_k))} \\
= \quad & (2^{a_j} - 1) 2^{-(a_j-1)-(k-1)} + \sum_{l=0}^{k-2} 2^{a_j-1} 2^{-(a_j-1)-l} \\
< \quad & 2^{-(k-2)} + \sum_{l=0}^{k-2} 2^{-l} = 2,
\end{aligned}$$

so by Theorem 3, this movement cannot be made. Hence no pebbles can be moved out of $Q_j \times P_m^*$. Similarly, in $(Q_1 \cup \{v\}) \times P_m^*$, Theorem 3 prevents a pebble from being moved to (v, p_n), so we cannot move a pebble to (v, p_n), and we are done. ∎

Theorem 7 *Let G be a graph of order m with diameter Δ. Then for $n > 3(2^\Delta - 1)m$,*

$$f(K_n \times G) = mn.$$

Proof. Let G be a graph of order m with diameter Δ, and let $n > 3(2^\Delta - 1)m$. For all graphs H, $f(H) \geq |V(H)|$, so it is clear that $f(K_n \times G) \geq mn$. Suppose there are at least mn pebbles on the vertices of $K_n \times G$. We will show that we can move a pebble to any $(y, v) \in V(K_n \times G)$. If $w \in V(G)$ exists with

$$\sum_{x \in V(K_n)} \left\lfloor \frac{\mathcal{N}(x, w)}{2} \right\rfloor \geq 2^\Delta, \tag{9}$$

we will be able to move 2^Δ pebbles to (y,w) and hence at least 1 pebble to (y,v). Hence we can assume that for all w,

$$2^\Delta - 1 \geq \sum_{x\in V(K_n)} \left\lfloor \frac{\mathcal{N}(x,w)}{2} \right\rfloor \geq \sum_{x\in V(K_n),\ \mathcal{N}(x,w)\geq 2} \frac{\mathcal{N}(x,w)-1}{2}.$$

However, if (9) does not hold, there can be at most $2^\Delta - 1$ x's with $\mathcal{N}(x,w) \geq 2$, so

$$\sum_{x\in V(K_n),\ \mathcal{N}(x,w)\geq 2} \mathcal{N}(x,w) \leq 2(2^\Delta - 1) + (2^\Delta - 1) = 3(2^\Delta - 1).$$

Then

$$\sum_{(x,w)\in V(K_n\times G),\ \mathcal{N}(x,w)\geq 2} \mathcal{N}(x,w) \leq 3(2^\Delta - 1)m,$$

so at least $mn - 3(2^\Delta - 1)m$ vertices of $K_n \times G$ have exactly 1 pebble on them, and at most $3(2^\Delta - 1)m$ do not. But $n > 3(2^\Delta - 1)m$, so there must be some $x \in V(K_n)$ such that every $\mathcal{N}(x,w) = 1$. Then since there are at least mn pebbles on $K_n \times G$, if there is not already a pebble on (y,v), there must be at least 2 pebbles on some vertex, $(\bar{x},\bar{w})$ say. Then we can move a pebble to $(x,\bar{w})$, and pebbling from (x,w_1) to (x,w_2), …, (x,w_{q-1}) to (x,w_q), where $(\bar{w} = w_1,\ldots,w_q = v)$ is a path in G, we get 2 pebbles on (x,v), so we can move a pebble to (y,v), as desired. ∎

Lemma 8 *If we have put pebbles on the vertices of P_n^* in such a way that*

$$\sum_{i=1}^{n} \mathcal{N}(p_i)2^{-(n-i)} \geq 1,$$

then we can pebble p_n.

Proof. By the same argument as in Theorem 4, whether or not we can pebble p_n will be unchanged if we take a pebble off p_j and put two pebbles on p_{j-1}. Doing this repeatedly, we get

$$\sum_{i=1}^{n} \mathcal{N}(p_i)2^{i-1}$$

pebbles on p_1. But by hypothesis, this number is at least 2^{n-1}, so we can clearly pebble p_n; hence we are done. ∎

Let P_m be the graph with $V(P_m) = \{p_1,\ldots,p_m\}$ and $E(P_m) = \{\{p_1,p_2\},\ldots,\{p_{m-1},p_m\}\}$.

Lemma 9 *For all vertices x of K_m, $m \geq 3$, we have*

$$f((p_{\Delta+1}, x), P^*_{\Delta+1} \times K_m) \leq 2^{\Delta+1} + 2m - 5 \tag{10}$$

for all sufficiently large Δ.

Proof. Let $V(K_m) = \{x = x_0, \ldots, x_{m-1}\}$. We induce on Δ. Suppose $f((p_\Delta, x_0), P^*_\Delta \times K_m) \leq 2^\Delta + r$. Then we will prove that, if Δ is sufficiently large, $f((p_{\Delta+1}, x_0), P^*_{\Delta+1} \times K_m) \leq 2^{\Delta+1} + \max(r-1, 2m-5)$. This will give (10) for (even larger) sufficiently large values of Δ.

Let $f((p_\Delta, x_0), P^*_\Delta \times K_m) \leq 2^\Delta + r$, and let there be pebbles on the vertices of $P^*_{\Delta+1} \times K_m$. Let $S_b = \{p_2, \ldots, p_{\Delta+1}\} \times K_m$, $S_t = \{p_1\} \times K_m$. Let p_b be the total number of pebbles in S_b, p_t be the total number in S_t, and define q_t, q_b similarly. Then if

$$p_b + \frac{p_t - q_t}{2} \geq 2^\Delta + r,$$

we can first move $(p_t - q_t)/2$ pebbles into S_b and then pebble $(p_{\Delta+1}, x_0)$ in S_b. Hence we can assume that

$$\begin{aligned} \frac{p_t + q_t}{2} &> 2^{\Delta+1} + \max(r-1, 2m-5) - 2^\Delta - r \\ &\geq 2^\Delta - 1, \end{aligned}$$

so $p_t + q_t > 2^{\Delta+1} - 2$ and hence $p_t + q_t \geq 2^{\Delta+1}$. Let

$$\alpha = q_t + q_b, \qquad \beta = p_t - q_t, \qquad \gamma = p_b - q_b.$$

Then $q_t \leq m$ so $\beta = p_t - q_t \geq 2^{\Delta+1} - 2m$. Also,

$$\alpha + \beta + \gamma = p_t + p_b \geq 2^{\Delta+1} + 2m - 5. \tag{11}$$

Let

$$Q = \sum_{k=1}^{\Delta+1} \mathcal{N}(p_k, x_0) 2^{k-1}.$$

Then we will show that we can move pebbles until $Q \geq 2^\Delta$, at which point we will be able to pebble $(p_{\Delta+1}, x_0)$, by Lemma 8. Let $T_j = P^*_{\Delta+1} \times \{x_j\}$. Then pebbling from S_t into T_0, we can get at least $\beta/2$ pebbles in T_0. Pebbling from S_b, we can get at least $\gamma/2$ pebbles in T_0, and since none of them are on (p_1, x_0), this increases Q by at least γ. For each $j = 0, \ldots, m-1$, let α_j be the number of vertices in $P^*_{\Delta+1} \times \{x_j\}$ with an odd number of pebbles, and let β_j be $2\lfloor \mathcal{N}(p_1, x_j)/2 \rfloor$. Then Q starts out with a value of at least $2^{\alpha_0} - 1$. Hence after this pebbling, we have

$$Q \geq \frac{\beta}{2} + \gamma + 2^{\alpha_0} - 1.$$

Now β is even, so let $\beta = 2^{\Delta+1} - 2\theta$, where $\theta \leq m$. Then $2^\Delta - Q \leq \theta - \gamma - (2^{\alpha_0} - 1)$, so if we do not already have $Q \geq 2^\Delta$, we must have $\theta \in \{1, \ldots, m\}$ and $\gamma \in \{0, \ldots, \theta - 1\}$. From (11), $\alpha \geq 2m - 5 + 2\theta - \gamma$. Now we have α_j pebbles left in T_j for $j = 1, \ldots, m-1$, with at most one pebble on each vertex. Our strategy for increasing Q will be to redirect some of the pebbles in S_t that were counted in β to pebble other T_j's instead of T_0, so that if we spend z pebbles from β in this manner, we can increase Q by more than the $z/2$ we would have originally.

Consider a T_j that has $\alpha_j = k \geq 3$ and pebbles on (p_{R_1}, x_j),. . .,(p_{R_k}, x_j), where $R_1 < R_2 < \ldots < R_k$. Now if we move $2^{R_k - 1} - 2^{R_{k-1}-1} - \ldots - 2^{R_1 - 1}$ pebbles onto (p_1, x_j) at the start, pebbling down T_j, we can move one pebble to (p_{R_k}, x_j), and then one pebble to (p_{R_k}, x_0). This increases Q by $2^{R_k - 1}$ and uses $2^{R_k} - 2^{R_{k-1}} - \ldots - 2^{R_1}$ pebbles from β. Hence it increases Q from the original estimate by $2^{R_{k-1}-1} + \ldots + 2^{R_1 - 1} = Y_j$, say, and uses $2^{R_k} - 2Y_j$ pebbles from β. Now if $Y_j \geq m$, we have enough pebbles in β to do this, since $\beta \geq 2^{\Delta+1} - 2m$, and furthermore, increasing Q by m raises it above 2^Δ, so we are done. If $Y_j < m$, then we can move $2^{R_{k-1}-1} - 2^{R_{k-2}-1} - \ldots - 2^{R_1 - 1}$ pebbles onto (p_1, x_j) at the start, pebble down T_j as before, and then pebble from $(p_{R_{k-1}}, x_j)$ to $(p_{R_{k-1}}, x_0)$. This increases Q from the original estimate by

$$2^{R_{k-2}-1} + \ldots + 2^{R_1 - 1} \geq 2^{k-3} + \ldots + 2^0 = 2^{k-2} - 1$$

and uses no more than $2Y_j \leq 2m$ pebbles from β. Then if

$$\sum_{j \neq 0} (2^{\alpha_j - 2} - 1) \geq \theta - \gamma - (2^{\alpha_0} - 1) \tag{12}$$

we will be done for Δ large enough so that $2^{\Delta+1} - 2m \geq 2m(m-1)$. This is because the only positive contributions to the left-hand side of (12) come from T_j's with $\alpha_j \geq 3$. Then either some $Y_j \geq m$ and we will be done, or all $Y_j < m$, so we will be able to increase Q by at least the left-hand side of (12), since $\beta \geq 2m(m-1)$, and we will be done. But

$$\begin{aligned}\sum_{j\neq 0}(2^{\alpha_j-2}-1)+(2^{\alpha_0}-1) &\geq \sum_{j\neq 0}(\alpha_j-2)+\alpha_0\\ &= \alpha-(2m-2)\\ &\geq 2m-5+2\theta-\gamma-(2m-2)\\ &= 2\theta-\gamma-3,\end{aligned}$$

so (12) will hold for $\theta \geq 3$.

Suppose $\theta = 1$. Then $\gamma = 0$. If $\alpha_0 \geq 1$, we already have $Q \geq 2^\Delta$. If $\alpha_0 = 0$, then, since $m \geq 3$, $\alpha \geq 2m-5+2\theta-\gamma = 2m-3 > m-1$. Hence there must be some $\alpha_j \geq 2$. Let $\alpha_j \geq 2$, and let there be pebbles at (p_k, x_j) and (p_l, x_j), where $k > l$. Now let us move $2^{k-1}-2^{l-1}$ pebbles onto (p_1, x_j) at the start. Then pebbling down T_j, we can move a pebble to (p_k, x_0). This increases Q by 2^{k-1} and uses $2^k - 2^l$ pebbles from β, and since $\beta = 2^{\Delta+1}-2$, β is large enough to do this. Then we have increased Q by $2^{l-1} \geq 1$ from its original estimate, so we are done.

Suppose $\theta = 2$. Then if $\gamma = 1$ and $\alpha_0 \geq 1$, we are done, and if $\alpha_0 \geq 2$, we are also done. Suppose either $\gamma = 1$ and $\alpha_0 = 0$ or $\gamma = 0$ and $\alpha_0 = 1$. Then $\alpha - \alpha_0 \geq 2m-2$. If there is some $\alpha_j \geq 3$, then the left-hand side of (12) will be at least 1, so we will be done. Otherwise, we must have $\alpha_j = 2$ for all $j \neq 0$. Then if $\beta_0 \geq 2^\Delta$, Q will be at least 2^Δ, so we will be done. Otherwise $\beta - \beta_0 > 2^\Delta - 4$, so there is $j \neq 0$ with $\beta_j > (2^\Delta-4)/(m-1)$. In particular, if Δ is large enough, $\beta_j \geq 2$. Then if there are pebbles in T_j at (p_k, x_j) and (p_l, x_j), where $k > l$, moving $2^{k-1}-2^{l-1}$ pebbles onto (p_1, x_j) takes at most $2^k - 2^l - 2$ pebbles from β, since we save 2 pebbles due to the 2 pebbles already on (p_1, x_j) (unless $2^k - 2^l < 4$, in which case $k = 2$ and $l = 1$, so that it takes just 1 pebble from β_j.) In any case, $\beta = 2^{\Delta+1}-4$, so we have enough pebbles to proceed as in the $\theta = 1$ case, and we can increase Q by at least 1 from its original estimate, so we are done.

Finally, suppose $\theta = 2$ and $\gamma = \alpha_0 = 0$. Then $\alpha - \alpha_0 \geq 2m-1$. If there is some j with $\alpha_j \geq 4$, or j and k with $\alpha_j \geq 3$ and $\alpha_k \geq 3$, then the left-hand side of (12) will be at least 2 so we will be done. Otherwise, we must have $\alpha_j = 3$ for some j and $\alpha_i = 2$ for all other $i \neq 0$. Then if there are pebbles in T_j at (p_k, x_j), (p_l, x_j), and (p_r, x_j), where $k > l > r$, moving $2^{k-1}-2^{l-1}-2^{r-1}$ pebbles onto (p_1, x_j) will use only $2^k - 2^l - 2^r$ pebbles from β; but this is no larger than $2^{\Delta+1}-6$, so β is large enough to do this. Then this increases Q by $2^{l-1}+2^{r-1}\geq 3$ from its original estimate, so we are done. ∎

Lemma 10 *Let G be a connected graph and let v be a vertex of G. Then there is an integer $K(v,G)$ such that for all Δ,*

$$f((p_{\Delta+1},v),P^*_{\Delta+1}\times G)\leq 2^{\Delta+h}+(\Delta+1)K(v,G),$$

where $h=\max_{w\in V(G)} d(w,v)$.

Proof. Let T be a spanning tree of G which preserves distances from v. Then if $(a_1,\ldots,a_n)$ is a path-size sequence of a maximum path-partition for T, $a_1=h$, so applying Theorem 6 to $T^*_v\times P^*_{\Delta+1}$, we can set $K(v,G)=\sum_{i=2}^{n}2^{a_i}-n+1$. ∎

Lemma 11 *Let G be a connected graph with diameter h. Then for all sufficiently large Δ,*

$$f(P_{\Delta+1}\times G)=\max_{w\in Q}(f((p_{\Delta+1},w),P_{\Delta+1}\times G)), \tag{13}$$

where Q is the set of vertices w of G such that there exists a vertex v of G with $d(v,w)=h$.

Proof. Choose vertices v and w of G such that $d(v,w)=h$. If we put $2^{\Delta+h}-1$ pebbles on (p_1,v), Theorem 3 shows that we cannot move a pebble to $(p_{\Delta+1},w)$. Hence the right-hand side of (13) is at least $2^{\Delta+h}$. Let $K=\max_{w\in V(G)}K(w,G)$, and let Δ be big enough so that $2^{\Delta-1+h}\geq 2^{h+1}+(\Delta+2)K$. Now let $j\in\{2,\ldots,\Delta\}$, and fix $v\in V(G)$. We will show that $f((p_j,v),P_{\Delta+1}\times G)\leq 2^{\Delta+h}$. If we set $S_1=\{p_1,\ldots,p_j\}\times G$ and $S_2=\{p_j,\ldots,p_{\Delta+1}\}\times G$, Lemma 10 implies that if there are at least $2^{j-1+h}+jK(v,G)$ pebbles on S_1, we can pebble (p_j,x), and if there are at least $2^{\Delta+1-j+h}+(\Delta+2-j)K(v,G)$ pebbles on S_2, we can also pebble (p_j,x). Hence to pebble (p_j,x) it will do to have at least $2^{j-1+h}+2^{\Delta+1-j+h}+(\Delta+2)K(v,G)$ pebbles on G. But for $j\in\{2,\ldots,\Delta\}$,

$$2^{j-1+h}+2^{\Delta+1-j+h}\leq 2^{h+1}+2^{\Delta-1+h},$$

so we only need to have $2^{h+1}+2^{\Delta-1+h}+(\Delta+2)K(v,G)$ pebbles on G, and this quantity is no larger than $2^{\Delta+h}$ by our assumption on Δ. Now

$f((p_1,v),P_{\Delta+1}\times G) = f((p_{\Delta+1},v),P_{\Delta+1}\times G)$ for all $v\in V(G)$. Together with the above, this shows that

$$f(P_{\Delta+1}\times G) = \max_{w\in V(G)} f((p_{\Delta+1},w),P_{\Delta+1}\times G).$$

Now let $v\in V(G)$ have no $w\in V(G)$ such that $d(w,v)=h$. Then we must have $d(w,v)\le h-1$ for all $w\in V(G)$, so by Lemma 10,

$$f((p_{\Delta+1},v),P_{\Delta+1}\times G)\le 2^{\Delta+h-1}+(\Delta+1)K(v,G),$$

and this is also no larger than $2^{\Delta+h}$. Hence we have (13), as desired. ∎

Theorem 12 *For all $m\ge 3$ and sufficiently large Δ,*

$$f(P_{\Delta+1}\times K_m) = 2^{\Delta+1}+2m-5.$$

Proof.

($\le$): This follows from Lemmas 9 and 11.

($\ge$): We will show that $f(P_{\Delta+1}\times K_m)\ge 2^{\Delta+1}+2m-5$ for all $\Delta\ge 1$. Let $V(K_m)=\{x_0,\ldots,x_{m-1}\}$. Put $2^{\Delta+1}-3$ pebbles on (p_1,x_0), 1 pebble on (p_1,x_j) for $j=1,\ldots,m-1$, and 1 pebble on $(p_{\Delta+1},x_j)$ for $j=2,\ldots,m-1$. This gives a total of $2^{\Delta+1}+2m-6$ pebbles. We will show that with this starting configuration of pebbles, we cannot pebble $(p_{\Delta+1},x_1)$. Suppose we did, in fact, have some sequence of moves that pebbled $(p_{\Delta+1},x_1)$. Consider the first pebbling move we make into $\{p_{\Delta+1}\}\times K_m$. If this move is onto $(p_{\Delta+1},x_j)$, where $1\le j\le m-1$, then 2 pebbles were on (p_Δ,x_j) prior to this move. Otherwise the move is to $(p_{\Delta+1},x_0)$, but putting a pebble on $(p_{\Delta+1},x_0)$ does not enable us to do any pebbling from the bottom row, so there must be some succeeding move into the bottom row. If the next move into the bottom row is onto $(p_{\Delta+1},x_j)$, where $1\le j\le m-1$, then 2 pebbles were on (p_Δ,x_j) just before this move, and there would still have been at least 2 pebbles on (p_Δ,x_j) prior to this move if we did not make the move to $(p_{\Delta+1},x_0)$. Otherwise the move must be to $(p_{\Delta+1},x_0)$, so if we omit the first move to $(p_{\Delta+1},x_0)$, we must have at least 4 pebbles on (p_Δ,x_0) before this move. This shows that by pebbling only in $\{p_1,\ldots,p_\Delta\}\times K_m$, we can either move 2 pebbles to (p_Δ, x_j), for some $j \in \{1, \ldots, m-1\}$, or move 4 pebbles to (p_Δ, x_0). But if we set

$$\begin{aligned} Q_i &= \sum_{j=1}^{\Delta} 2^{j-1}\mathcal{N}(p_j, x_i), \qquad i = 0, \ldots, m-1, \text{ and} \\ Q &= \sum_{i=1}^{m-1} \max(\frac{Q_i - 1}{2}, 0) + \frac{1}{2}\left\lceil \frac{Q_0 - 1}{2} \right\rceil \end{aligned}$$

we would then have $Q \geq 2^{\Delta-1} - \frac{1}{2}$. Now since Q_i does not increase when we pebble from some (x_i, p_j) to (x_i, p_{j-1}) or (x_i, p_{j+1}), Q does not increase either. But if we pebble from (x_i, p_j) to (x_k, p_j), where $i, k > 0$, then for some $r > 0$, Q_i will decrease by $2r$ and Q_k will increase by r; then $\max((Q_i - 1)/2, 0)$ will decrease by at least $(2r-1)/2$ and $\max((Q_k - 1)/2, 0)$ will increase by at most $r/2$, so the net change in Q is no larger than $(1-r)/2 \leq 0$. If we pebble from (x_0, p_j) to (x_i, p_j), $i > 0$, and Q_0 decreases by $2r$ while Q_i increases by r, then $(1/2)\lceil (Q_0 - 1)/2 \rceil$ will decrease by $r/2$ and $\max((Q_i - 1)/2, 0)$ will increase by at most $r/2$, so Q does not increase. Finally, if we pebble from (x_i, p_j) to (x_0, p_j), $i > 0$, and Q_i decreases by $2r$ while Q_0 increases by r, then $(1/2)\lceil (Q_0 - 1)/2 \rceil$ will increase by at most $r/2$ while $\max((Q_i - 1)/2, 0)$ will decrease by at least $(2r-1)/2$, so the net change in Q is no larger than $(1-r)/2 \leq 0$. So Q never increases; but the initial value of Q is only $2^{\Delta-1} - 1$, so we have a contradiction, as desired. ∎

Lemma 13 *Suppose there are pebbles on $P^*_{\Delta+1} \times P_{r+1}$, and let $\mathcal{N}(p_1, p_j) = \epsilon_j$ and $\sum_{i=1}^{\Delta+1} 2^{i-1}\mathcal{N}(p_i, p_j) = w_j$ for $j = 1, \ldots, r+1$. Suppose $\epsilon_1 + \ldots + \epsilon_r + \epsilon_{r+1}/2 \geq 2^{\Delta+r} - Q$, where $Q \leq 2^{\Delta-1} - 2^{r-1}$. As well as the usual pebbling operations, suppose that we either have that*

1. *We can take $2^k - R$ pebbles off (p_1, p_l) and put 1 pebble on (p_{k+1}, p_l), for some fixed l and k with $1 \leq l \leq r$, $1 \leq k \leq \Delta$, where $R - Q \geq 2^r$, or*

2. *For each $j = 1, \ldots, r$, we can take $P_j \leq S = 2^{\Delta-r-1}$ pebbles off (p_1, p_j) and distribute these pebbles on column j into a configuration with $\sum_{i=1}^{\Delta+1} 2^{i-1}\mathcal{N}(p_i, p_j) = R_j$, where $w_{r+1} + \sum_{j=1}^{r}(w_j + R_j - P_j) \geq 2^{\Delta+r}$.*

Then we can pebble $(p_{\Delta+1}, p_{r+1})$, and furthermore, if in 1. we have $\epsilon_l \geq 2^k - R$, or if in 2. we have some $\epsilon_j \geq P_j$, then we can pebble $(p_{\Delta+1}, p_{r+1})$

performing the special operation in 1. at the beginning of our pebbling, or performing the special operation in 2. on $P^*_{\Delta+1} \times \{p_j\}$ at the beginning of our pebbling.

Proof. First, move $\lfloor \epsilon_{r+1}/2 \rfloor$ pebbles onto (p_1, p_r) from (p_1, p_{r+1}). Then we have $\epsilon_1 + \ldots + \epsilon_r \geq 2^{\Delta+r} - Q$. Now we have 2 cases corresponding to 1. and 2. in the statement of the lemma.

Case 1.: For each $j = 1, \ldots, r$, we do the following, in order:

Step (1). If $j = l$, expend $2^k - R$ pebbles from (p_1, p_l) to put a pebble on (p_{k+1}, p_l).

Step (2). If $j \geq l$, expend 2^k pebbles from (p_1, p_l) to move a pebble to (p_{k+1}, p_j). This gives 2 pebbles on (p_{k+1}, p_j), so move a pebble to (p_{k+1}, p_{j+1}).

Step (3). Pebble everything possible from (p_1, p_j) to (p_1, p_{j+1}).

Let γ_j be the number of pebbles on (p_1, p_j) before step (1). Let

$$v_j = \begin{cases} 0, & j < l \\ 2^{k+1} - R, & j = l \\ 2^k, & j > l. \end{cases}$$

Then for steps (1)-(3) to be possible, we need to have $\gamma_j \geq v_j$ for $j = l, \ldots, r$, and we will have

$$\gamma_{j+1} = \left\lfloor \frac{\gamma_j - v_j}{2} \right\rfloor + \epsilon_{j+1} \geq \frac{\gamma_j - v_j - 1}{2} + \epsilon_{j+1},$$

$j = 1, \ldots, r$. Then since $\gamma_1 = \epsilon_1$, we have

$$\gamma_j \geq \sum_{i=1}^{j} \epsilon_i 2^{-(j-i)} - \sum_{i=1}^{j-1} (v_i + 1) 2^{-(j-i)}.$$

Now if $\epsilon_{j+1} + \ldots + \epsilon_r \geq 2^{\Delta+1+r-(j+1)}$, there are at least $2^{\Delta+1+r-(j+1)}$ pebbles in $P^*_{\Delta+1} \times \{p_{j+1}, \ldots, p_{r+1}\}$, so we can pebble $(p_{\Delta+1}, p_{r+1})$ within $P^*_{\Delta+1} \times \{p_{j+1}, \ldots, p_{r+1}\}$, by Theorem 6. Hence we can assume that

$$\epsilon_1 + \ldots + \epsilon_j \geq 2^{\Delta+r} - Q - 2^{\Delta+r-j}$$

except that

$$\epsilon_1 + \ldots + \epsilon_r \geq 2^{\Delta+r} - Q,$$

so

$$\sum_{i=1}^{j} \epsilon_i 2^{-(j-i)} \geq 2^{-(j-1)}(2^{\Delta+r} - Q - 2^{\Delta+r-j}(1-\delta_{jr}))$$

where δ is the Kronecker delta function. Then to satisfy $\gamma_j \geq v_j$ for $j = l, \ldots, r$, we need to have

$$2^{-(j-1)}(2^{\Delta+r} - Q - 2^{\Delta+r-j}(1-\delta_{jr})) \geq \sum_{i=1}^{j-1}(v_i+1)2^{-(j-i)} + v_j,$$

but

$$\begin{aligned}\sum_{i=1}^{j-1}(v_i+1)2^{-(j-i)} + v_j &= (2^{-1} + \ldots + 2^{-(j-1)}) + (2^{k+1} - R)2^{-(j-l)} \\ &\quad +2^k(2^{-(j-l-1)} + \ldots + 1) \\ &\leq 1 + 2^{k+1} - R2^{-(j-l)}.\end{aligned}$$

Then

$$R2^{-(j-l)} - Q2^{-(j-1)} \geq (R-Q)2^{-(j-1)} \geq 2^r 2^{-(j-1)} \geq 1,$$

so it will do to have

$$2^{\Delta+r-(j-1)} \geq 2^{k+1} + 2^{\Delta+r-(2j-1)}(1-\delta_{jr}),$$

or, since $k \leq \Delta$, it will do to show that

$$2^{\Delta+r-(j-1)} \geq 2^{\Delta+1} + 2^{\Delta+r-(2j-1)}(1-\delta_{jr}). \tag{14}$$

If $j = r$, this reduces to $2^{\Delta+1} \geq 2^{\Delta+1}$. Otherwise, $\Delta + 1 < \Delta + r - (j-1)$ and $\Delta + r - (2j-1) < \Delta + r - (j-1)$, so (14) holds, as desired. Hence we can always perform the pebbling program outlined. After doing it, we will have γ_{r+1} pebbles on (p_1, p_{r+1}), but

$$\begin{aligned}\gamma_{r+1} &\geq \sum_{i=1}^{r+1} \epsilon_i 2^{-(r+1-i)} - \sum_{i=1}^{r}(v_i+1)2^{-(r+1-i)} \\ &\geq 2^{-r}(2^{\Delta+r} - Q) - (1 + 2^k - R2^{-(r+1-l)}) \\ &\geq 2^{\Delta} + 2^{-r}(R-Q) - 1 - 2^k \\ &\geq 2^{\Delta} - 2^k.\end{aligned}$$

But we also have a pebble on (p_{k+1}, p_{r+1}), so by Lemma 8, we can pebble $(p_{\Delta+1}, p_{k+1})$, as desired. Also, from above, it is clear that if $\epsilon_l \geq 2^k - R$, we can perform operation 1. first, as desired.

Case 2.: For each $j = 1, \ldots, r$, we do the following, in order:

Step (1). Take P_j pebbles off (p_1, p_j) and redistribute them.

Step (2). Pebble from (p_2, p_j) to (p_2, p_{j+1}), ..., $(p_{\Delta+1}, p_j)$ to $(p_{\Delta+1}, p_{j+1})$ so that there is no more than one pebble left on each of (p_2, p_j), ..., $(p_{\Delta+1}, p_j)$.

Step (3). Suppose there are pebbles left on (p_{R_1}, p_j),...,(p_{R_k}, p_j), where $1 < R_1 < \ldots < R_k$. Then remove pebbles from (p_1, p_j) and pebble down $P^*_{\Delta+1} \times \{p_j\}$ until there are two pebbles on (p_{R_k}, p_j) and no pebbles on (p_2, p_j),...,$(p_{R_k - 1}, p_j)$. This takes

$$2^{R_k - 1} - 2^{R_{k-1} - 1} - \ldots - 2^{R_1 - 1} \leq 2^{\Delta}$$

pebbles.

Step (4). Pebble from (p_{R_k}, p_j) to (p_{R_k}, p_{j+1}).

Step (5). Pebble everything possible from (p_1, p_j) to (p_1, p_{j+1}).

Let γ_j be as before. Then for steps (1)-(3) to be possible, the criterion is the same as in case 1., except that we have $v_j = 2^{\Delta} + S$ for $j = 1, \ldots, r$. Hence we need, for $j = 1, \ldots, r$,

$$\begin{aligned} 2^{-(j-1)}(2^{\Delta+r} - Q - 2^{\Delta+r-j}(1 - \delta_{jr})) &\geq \sum_{i=1}^{j-1}(v_i + 1)2^{-(j-i)} + v_j \\ &= (1 - 2^{-(j-1)}) + (2 - 2^{-(j-1)})(2^{\Delta} + S) \end{aligned}$$

so it will do to have

$$2^{\Delta+r-(j-1)} - 2^{\Delta+r-(2j-1)}(1 - \delta_{jr}) \geq 2^{-(j-1)}Q + 1 + (2 - 2^{-(j-1)})2^{\Delta} + 2S.$$

Then

$$1 + 2^{-(j-1)}Q + 2S \leq 2^{\Delta-j} + 2^{\Delta-r} + 1 - 2^{r-1}2^{-(j-1)} \leq 2^{\Delta-j+1},$$

so we need to have

$$2^{\Delta+r-(j-1)} - 2^{\Delta+r-(2j-1)}(1-\delta_{jr}) \geq 2^{\Delta+1}.$$

But this is (14) in case 1., which has already been proven. Hence we can carry out our pebbling procedure.

Now in our pebbling procedure, step (1) increases w_j by $R_j - P_j$, and steps (2)-(5) have the net effect of increasing w_{j+1} by at least $(w_j - 1)/2$, since we leave behind at most 1 pebble on (p_1, p_j) and no pebbles on any other (p_i, p_j). Let $\bar{w}_j$ be the initial value of w_j. Then after completing our procedure, we have

$$\begin{aligned} w_{r+1} &\geq \bar{w}_{r+1} + \sum_{j=1}^{r} 2^{-(r+1-j)}(\bar{w}_j + R_j - P_j - 1) \\ &\geq 2^{-r}(\sum_{j=1}^{r}(\bar{w}_j + R_j - P_j) + \bar{w}_{r+1}) - (1 - 2^{-r}) \\ &\geq 2^{-r}2^{\Delta+r} - (1-2^{-r}) = 2^{\Delta} - (1 - 2^{-r}) \end{aligned}$$

but since w_{r+1} is an integer, we have $w_{r+1} \geq 2^{\Delta}$, and by Lemma 8, we can pebble $(p_{\Delta+1}, p_{r+1})$, as desired; also, if some $\epsilon_j \geq P_j$, we can clearly perform operation 2. on $P^*_{\Delta+1} \times \{p_j\}$ at first, as desired. ∎

A tree T is a *caterpillar* if there exists a path P in T such that no vertex in T has distance greater than 1 from P. A maximum-length path in a caterpillar T is a *backbone* for T. The backbone of T is unique, up to a choice of initial and final vertex, so all backbones for T have the same number of vertices. The vertices not in a backbone are called *legs*; there can be no legs adjacent to the end-vertices of a backbone, for then the backbone would not have maximum length. Note that this implies that a caterpillar with a backbone of 2 vertices must be just P_2.

Lemma 14 *Let C be a caterpillar with m legs and a backbone B with $r+2$ vertices. Then for all end-vertices x of B and sufficiently large Δ, we have*

$$f((p_{\Delta+1}, x), P^*_{\Delta+1} \times C) \leq 2^{\Delta+r+1} + \left\lceil \frac{\Delta+2}{2} \right\rceil m.$$

Proof. Let $V(B) = \{x_{r+2}, x_1, \ldots, x_{r+1} = x\}$ and $E(B) = \{\{x_{r+2}, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_r, x_{r+1}\}\}$. Let the legs of C be $x_{r+3}, \ldots, x_{m+r+2}$. If $r = 0$, we must have $C = P_2$, so our result follows from Theorem 6. Otherwise let $r \geq 1$. We induct on Δ as in the proof of Lemma 9. Define r, S_b, S_t, p_t, p_b, q_t, q_b, α, β, γ, α_j, β_j, and T_j, for $j = 1, \ldots, m+r+2$, as in Lemma 9, substituting the graph C for the graph K_m. Then we have $q_t \leq |V(C)| = m+r+2$ and as in Lemma 9, we can assume that $\beta \geq 2^{\Delta+r+1} - 2(m+r+2)$. Also,

$$\alpha + \beta + \gamma = p_t + p_b \geq 2^{\Delta+r+1} + \left\lceil \frac{\Delta+2}{2} \right\rceil m.$$

Let

$$X = \sum_{i=1}^{r+1} \sum_{j=1}^{\Delta+1} \mathcal{N}(p_j, x_i) 2^{j-1}.$$

Then at the start, X is at least

$$\sum_{i=1}^{r+1} (\beta_i + 2^{\alpha_i} - 1) \geq \sum_{i=1}^{r+1} (\alpha_i + \beta_i),$$

and pebbling into $\cup_{i=1}^{r+1} T_i = T$, say, gives

$$X \geq \sum_{i=1}^{r+1} (\alpha_i + \beta_i) + \sum_{i=r+2}^{m+r+2} \frac{\beta_i}{2} + \gamma.$$

Now if $\epsilon_j = \mathcal{N}(p_1, x_j)$ after this pebbling, $j = 1, \ldots, r+1$, we have

$$\epsilon_{r+1} \geq \beta_{r+1}, \qquad \sum_{i=1}^{r} \epsilon_i \geq \sum_{i=1}^{r} \beta_i + \sum_{i=r+2}^{m+r+2} \frac{\beta_i}{2},$$

so $\epsilon_1 + \ldots + \epsilon_r + \epsilon_{r+1}/2 \geq \beta/2 \geq 2^{\Delta+r} - (m+r+2)$, so if we set $Q = m+r+2$, and Δ is big enough so that $Q \leq 2^{\Delta-1} - 2^{r-1}$, the first hypothesis of Lemma 13 will be satisfied. Then

$$2^{\Delta+r} - X \leq 2^{\Delta+r} - \frac{\beta}{2} - \gamma - \sum_{i=1}^{r+1} \alpha_i.$$

If $X \geq 2^{\Delta+r}$, Lemma 13 says that we are done already (using condition 2. and setting all P_j=R_j=0). Otherwise, our strategy for pebbling $(p_{\Delta+1}, x_{r+1})$

will be to find pebbling moves out of and back into T, and alternate moves to take before the initial pebbling into T, such that either condition 1. or condition 2. of Lemma 13 is satisfied. Satisfying condition 2. of Lemma 13 will involve increasing X to at least $2^{\Delta+r}$.

Let $j \in \{r+2, \ldots, m+r+2\}$, and let $k = \alpha_j$. Also let x_j be adjacent to x_t in C, where $t \in \{1, \ldots, r\}$, and let there be pebbles left in T_j at (p_{R_1}, x_j), …, (p_{R_k}, x_j), where $R_1 < R_2 < \ldots < R_k$. Then for $l \in \{1, \ldots, k-1\}$ with $R_{l+1} = R_l + 1$, suppose that we move

$$2^{R_l - 1} - 2^{R_{l-1}-1} - \ldots - 2^{R_1 - 1}$$

pebbles onto (p_1, x_j). Then we can move one pebble to (p_{R_l+1}, x_j) and then one pebble to (p_{R_l+1}, x_t), increasing X by 2^{R_l}. We did this at the cost of decreasing β_t by

$$2^{R_l} - 2^{R_{l-1}} - \ldots - 2^{R_1},$$

so we have a net increase to X of $2^{R_{l-1}} + \ldots + 2^{R_1}$. Suppose that $k \geq (\Delta+2)/2$. Then if $R_l > \Delta - r - m - 1$, we must have

$$l \geq k - (\Delta + 1 - R_l) > \frac{\Delta+2}{2} - (\Delta+1) + (\Delta - r - m - 1) = \frac{\Delta}{2} - r - m - 1.$$

Hence for big enough Δ, $2^l - 2 \geq Q + 2^r$, and since

$$\begin{aligned} 2^{R_l} - 2^{R_{l-1}} - \ldots - 2^{R_1} &\leq 2^{R_l} - (2^{l-1} + \ldots + 2) \\ &= 2^{R_l} - (2^l - 2), \end{aligned}$$

condition 1. of Lemma 13 is satisfied and we are done. Otherwise $R_l \leq \Delta - r - m - 1$. Then we have to move at most $2^{\Delta-r-m-1} \leq 2^{\Delta-r-1}/(m+1)$ pebbles from β_t, and by doing this we can increase X by

$$2^{R_{l-1}} + \ldots + 2^{R_1} \geq 2^{l-1} + \ldots + 2^1 = 2^l - 2.$$

Let y be maximal with $R_y + 1 = R_{y+1}$. Then there can be at most $\lfloor (\Delta + 1 - R_{y+1})/2 \rfloor$ l's with $R_l > R_{y+1}$, so

$$(y+1) + \left\lfloor \frac{\Delta + 1 - R_{y+1}}{2} \right\rfloor \geq k$$

and since $R_y \geq y$,

$$k \leq y + 1 + \left\lfloor \frac{\Delta - y}{2} \right\rfloor \leq 1 + \frac{\Delta + y}{2},$$

so $y \geq 2k - \Delta - 2$, and we can increase X by at least $\max(2^{2k-\Delta-2} - 2, 0)$, under the assumption that $k \geq (\Delta + 2)/2$. But if $k < (\Delta + 2)/2$, then this is vacuously true, so for all $j \in \{r+2, \ldots, m+r+2\}$, we can increase X by at least

$$\max(2^{2\alpha_j - \Delta - 2} - 2, 0). \tag{15}$$

If there is $j \in \{r+2, \ldots, m+r+2\}$ with $\alpha_j \leq 1$, let w be such a j. Otherwise let $\alpha_j \geq 2$ for $j = r+2, \ldots, m+r+2$. We wish to show that for some $w \in \{r+2, \ldots, m+r+2\}$, we can increase X by at least $\alpha_w - 1$. Fix $j \in \{r+2, \ldots, m+r+2\}$, and suppose x_j is adjacent to x_q in C, where $q \in \{1, \ldots, r\}$. Let $\alpha_j = k$. Then if there are pebbles at (p_{R_1}, x_j), ..., (p_{R_k}, x_j), where $R_1 < \ldots < R_k$, and if, for some $l \in \{2, \ldots, k\}$,

$$\beta_j \geq 2^{R_l - 1} - \ldots - 2^{R_2 - 1} - 2^{R_1 - 1}(1 - \delta_{1R_1}),$$

we can pebble down T_j and move a pebble to (p_{R_l}, x_j) and then pebble (p_{R_l}, x_q). This increases X by $2^{R_l - 1}$ as compared to the increase we would obtain otherwise of

$$2^{R_l - 2} - \ldots - 2^{R_2 - 2} - 2^{R_1 - 2}(1 - \delta_{1R_1}),$$

so there is a net increase in X of

$$2^{R_l - 2} + \ldots + 2^{R_2 - 2} + 2^{R_1 - 2}(1 - \delta_{1R_1}) \quad \geq \quad 2^{l-2} + \ldots + 1 = 2^{l-1} - 1.$$

Since $2^{k-1} \geq k$, we have $\lceil \log_2 k \rceil + 1 \leq k$, so let $l = \lceil \log_2 k \rceil + 1$. Then

$$R_l \leq (\Delta + 1) - (k - l) = \Delta + 2 - k + \lceil \log_2 k \rceil$$

and if

$$\beta_j \geq 2^{\Delta + 1 - k + \lceil \log_2 k \rceil} \geq 2^{R_l - 1}$$

we can increase X by at least $2^{l-1} - 1 \geq k - 1$.

Now if

$$\beta_j < 2^{\Delta + 1 - \alpha_j + \lceil \log_2 \alpha_j \rceil} \qquad \text{for all } j \in \{r+2, \ldots, m+r+2\},$$

we have

$$\sum_{j=r+2}^{m+r+2} \beta_j < 2^{\Delta+1} \left(\sum_{j=r+2}^{m+r+2} 2^{-\alpha_j + \lceil \log_2 \alpha_j \rceil} \right).$$

Now for integer $x \geq 1$, $\lceil \log_2 x \rceil \leq (x+1)/2$. Hence

$$\sum_{j=r+2}^{m+r+2} \beta_j < 2^{\Delta+\frac{3}{2}} (\sum_{j=r+2}^{m+r+2} 2^{-\alpha_j/2}).$$

Now from (15), if some α_j has

$$2^{2\alpha_j-\Delta-2} - 2 \geq m + r + 2$$

we will be done, since we will be able to increase X above $2^{\Delta+r}$ and apply Lemma 13. But then for $j = r+2, \ldots, m+r+2$, we can assume that

$$2^{2\alpha_j-\Delta-2} < m + r + 4,$$

so for all j,

$$\alpha_j < \frac{\Delta + 2 + \log_2(m+r+4)}{2}.$$

Now

$$\begin{aligned} \sum_{j=r+2}^{m+r+2} \alpha_j &\geq 2^{\Delta+r+1} + \left\lceil \frac{\Delta+2}{2} \right\rceil m - \sum_{i=1}^{r+1} \alpha_i - \beta - \gamma \\ &\geq \left\lceil \frac{\Delta+2}{2} \right\rceil m + 2(2^{\Delta+r} - X). \end{aligned}$$

Hence if $X < 2^{\Delta+r}$, we must have $\sum_{j=r+2}^{m+r+2} \alpha_j \geq \left\lceil \frac{\Delta+2}{2} \right\rceil m$. For sufficiently large Δ, then,

$$8 + (m-1)\left(\frac{\Delta+2+\log_2(m+r+4)}{2}\right) \leq m\left(\frac{\Delta+2}{2}\right),$$

so $(\alpha_{r+2}, \ldots, \alpha_{m+r+2})$ is majorized by

$$(2, Y, \frac{\Delta+2+\log_2(m+r+4)}{2}, \ldots, \frac{\Delta+2+\log_2(m+r+4)}{2}),$$

where $Y \geq 6$. Hence

$$\begin{aligned} \sum_{j=r+2}^{m+r+2} 2^{-\alpha_j/2} &\leq 2^{-1} + 2^{-Y/2} + (m-1)2^{-(\Delta+2+\log_2(m+r+4))/4} \\ &\leq \frac{5}{8} + (m-1)2^{-(\Delta+2+\log_2(m+r+4))/4} \\ &\leq \frac{0.99}{\sqrt{2}}, \qquad \text{for sufficiently large } \Delta, \end{aligned}$$

so then

$$\begin{aligned}\sum_{j=1}^{r+1}\beta_j &= \beta - \sum_{j=r+2}^{m+r+2}\beta_j \\ &\geq 2^{\Delta+r+1} - 2(m+r+2) - 2^{\Delta+\frac{3}{2}}\frac{0.99}{\sqrt{2}} \\ &\geq 2^{\Delta+r} + 0.01\cdot 2^{\Delta+1} - 2(m+r+2), \qquad \text{since } r\geq 1 \\ &\geq 2^{\Delta+r}, \qquad \text{for sufficiently large } \Delta.\end{aligned}$$

Then by Lemma 13 with condition 2. and $P_j = R_j = 0$ for all j, we are done. Hence we can assume that some $j = w$, say, has $\beta_j \geq 2^{\Delta+1-\alpha_j+\lceil\log_2\alpha_j\rceil}$. Now decreasing β_j by x is equivalent, for the purposes of Lemma 13, to decreasing β_q by $x/2$, so if

$$\frac{1}{2}\left(2^{R_l-1} - \ldots - 2^{R_2-1} - 2^{R_1-1}(1-\delta_{1R_1})\right) \leq 2^{R_l-1} - (Q+2^r),$$

we can invoke condition 1. of Lemma 13, along with the fact that we can perform the special operation first. Otherwise,

$$2^{R_l-2} - \ldots - 2^{R_2-2} - 2^{R_1-2}(1-\delta_{1R_1}) > 2^{R_l-1} - (m+r+2+2^r),$$

so

$$\begin{aligned}(m+r+2)+2^r &> 2^{R_l-2} + \ldots + 2^{R_2-2} + 2^{R_1-2}(1-\delta_{1R_1}) \\ &\geq \frac{1}{2}\left(2^{R_l-1} - \ldots - 2^{R_2-1} - 2^{R_1-1}(1-\delta_{1R_1})\right)\end{aligned}$$

and if Δ is large enough, $2^{\Delta-r-1} \geq (m+r+2+2^r)(m+1)$, so the cost to β_q will not exceed $2^{\Delta-r-1}/(m+1)$.

Now we show that we can increase X by a total of

$$(\alpha_w - 1) + 2\left(\sum_{j=r+2,\ j\neq w}^{m+r+2}\alpha_j\right) - 2m\left\lceil\frac{\Delta+2}{2}\right\rceil.$$

If Δ is odd, $\lceil(\Delta+2)/2\rceil = (\Delta+3)/2$, and

$$\sum_{j=r+2,\ j\neq w}^{m+r+2}\max(2^{2\alpha_j-\Delta-2}-2, 0) \geq \sum_{j=r+2,\ j\neq w}^{m+r+2}(2^{2\alpha_j-\Delta-2}-2)$$

$$\geq \sum_{j=r+2,\ j\neq w}^{m+r+2} (2\alpha_j - \Delta - 3)$$

$$= 2\left(\sum_{j=r+2,\ j\neq w}^{m+r+2} \alpha_j\right) - 2m\frac{\Delta+3}{2},$$

as desired. If Δ is even, $\lceil(\Delta+2)/2\rceil = (\Delta+2)/2$. For even x, $\max(2^x - 2, 0) \geq x$, so

$$\sum_{j=r+2,\ j\neq w}^{m+r+2} \max(2^{2\alpha_j - \Delta - 2} - 2, 0) \geq \sum_{j=r+2,\ j\neq w}^{m+r+2} (2\alpha_j - \Delta - 2)$$

$$= 2\left(\sum_{j=r+2,\ j\neq w}^{m+r+2} \alpha_j\right) - 2m\frac{\Delta+2}{2},$$

as desired.

Now

$$\sum_{j=r+2,\ j\neq w}^{m+r+2} \alpha_j \geq \left\lceil \frac{\Delta+2}{2} \right\rceil m + 2(2^{\Delta+r} - X) - \alpha_w$$

so we can increase X by at least

$$4(2^{\Delta+r} - X) - \alpha_w - 1.$$

But if $2^{\Delta+r} - X \leq \alpha_w - 1$, we can increase X by $\alpha_w - 1$, making $X \geq 2^{\Delta+r}$, so we are done. Otherwise we can assume $\alpha_w - 1 < 2^{\Delta+r} - X$ so $\alpha_w + 1 \leq 3(2^{\Delta+r} - X)$, and we can increase X by at least $2^{\Delta+r} - X$. Furthermore, if we do not satisfy condition 1. of Lemma 13, we can do so at a cost of Y_1 pebbles from β_{a_1}, …, Y_{m+1} pebbles from $\beta_{a_{m+1}}$, where each a_j is in $\{1, \ldots, r\}$ and all $Y_j \leq 2^{\Delta-r-1}/(m+1)$. Hence condition 2. of Lemma 13 is then satisfied, so we are done. ∎

Theorem 15 *Let C be a caterpillar with m legs and a backbone with $r+2$ vertices. Then for all sufficiently large Δ,*

$$f(P_{\Delta+1} \times C) = 2^{\Delta+r+1} + \left\lceil \frac{\Delta+2}{2} \right\rceil m.$$

Proof.

($\leq$): C has diameter $r+1$, and vertices v and w of C have $d(v,w) = r+1$ iff v and w are distinct end-vertices of some backbone. Hence Lemma 14 tells us that for every vertex v of C such that a vertex w of C exists with $d(w,v) = r+1$,

$$f((p_{\Delta+1}, v), P_{\Delta+1} \times C) \leq 2^{\Delta+r+1} + \left\lceil \frac{\Delta+2}{2} \right\rceil m.$$

We can then apply Lemma 11 to get the desired result.

($\geq$): We will show that $f(P_{\Delta+1} \times C) \geq 2^{\Delta+r+1} + \left\lceil \frac{\Delta+2}{2} \right\rceil m$ for all $\Delta \geq 0$. Let B be a backbone of C, $V(B) = \{x_1, \ldots, x_{r+1}, x_{r+2}\}$, and $E(B) = \{\{x_1, x_2\}, \ldots, \{x_{r+1}, x_{r+2}\}\}$. Let the legs of C be $x_{r+3}, \ldots, x_{m+r+2}$. Put $2^{\Delta+r+1} - 1$ pebbles on (p_1, x_1), and 1 pebble on (p_j, x_i) for $i = r+3, \ldots, m+r+2$ and $j = 1, 2, 4, \ldots, 2\lfloor (\Delta+1)/2 \rfloor$. Then there are

$$2^{\Delta+r+1} - 1 + m\left(\left\lfloor \frac{\Delta+1}{2} \right\rfloor + 1\right) = 2^{\Delta+r+1} - 1 + \left\lceil \frac{\Delta+2}{2} \right\rceil m$$

pebbles on $P_{\Delta+1} \times C$. We will show that we cannot pebble $(p_{\Delta+1}, x_{r+2})$. Let

$$\begin{aligned} Q_i &= \sum_{j=1}^{\Delta+1} 2^{j-1} \mathcal{N}(p_j, x_i), \qquad i \in \{1, \ldots, m+r+2\}, \\ Q &= \sum_{i=1}^{r+2} 2^{i-1} Q_i. \end{aligned}$$

Then Q starts out equal to $2^{\Delta+r+1}-1$, and if there was a pebble on $(p_{\Delta+1}, x_{r+2})$, Q would be at least $2^{\Delta+r+1}$. But Q does not increase when we pebble within $P_{\Delta+1} \times B$. We need to show that it does not increase at other times either. Fix $i \in \{r+3, \ldots, m+r+2\}$, and let x_i be adjacent to x_q, where $q \in \{1, \ldots, r+2\}$. Now if we consider all the pebbling moves from $P_{\Delta+1} \times \{x_q\}$ into $P_{\Delta+1} \times \{x_i\}$, they must decrease Q_q by $2r$ and increase Q_i by r, for some $r \geq 0$. But if we increase Q_i by r, where

$$r < 2^{2j+1} - (1 + 2 + 8 + \ldots + 2^{2j-1}), \qquad j \geq 0,$$

we will not be able to move a pebble to (p_{2j+2}, x_i) afterwards. This is because (p_{2j+2}, x_i), …, $(p_{\Delta+1}, x_i)$ each start with at most 1 pebble, and hence we can make no pebbling moves out of (p_{2j+2}, x_i),. . .,$(p_{\Delta+1}, x_i)$ until we pebble (p_{2j+2}, x_i), which will put 2 pebbles on (p_{2j+2}, x_i). Hence if we could pebble (p_{2j+2}, x_i), by Theorem 3, we would have

$$\sum_{k=1}^{2j+2} \mathcal{N}(p_k, x_i) \geq 2 \cdot 2^{2j+1},$$

but

$$\sum_{k=1}^{2j+2} \mathcal{N}(p_k, x_i) \leq r + 1 + 2 + 8 + \ldots + 2^{2j+1} < 2 \cdot 2^{2j+1},$$

so this is impossible. But if we cannot pebble (p_{2j+2}, x_i), then in moving pebbles out of $P_{\Delta+1} \times \{x_i\}$, we can only move out of (p_1, x_i),. . .,(p_{2j+1}, x_i), so we can decrease Q_i by at most $r + 1 + 2 + 8 + \ldots + 2^{2j-1}$ and increase Q_q by at most $(r + 1 + 2 + 8 + \ldots + 2^{2j-1})/2$. Then the net change in Q_q is no bigger than

$$-2r + \frac{r + 1 + 2 + 8 + \ldots + 2^{2j-1}}{2} \quad = \quad -\frac{3r}{2} + \frac{1 + 2 + 8 + \ldots + 2^{2j-1}}{2}.$$

Now let $j \geq 0$ be minimal with $r < 2^{2j+1} - (1 + 2 + 8 + \ldots + 2^{2j-1})$. Then if $j = 0$, we have $r < 1$, so $r = 0$ and we clearly do not increase Q_q, since we can move no pebbles out of $P_{\Delta+1} \times \{x_i\}$. Otherwise,

$$r \geq 2^{2j-1} - (1 + 2 + 8 + \ldots + 2^{2j-3}),$$

so

$$\begin{aligned} & -\frac{3r}{2} + \frac{1}{2}(1 + 2 + 8 + \ldots + 2^{2j-1}) \\ \leq \;& -\frac{3}{2}(2^{2j-1} - (1 + 2 + 8 + \ldots + 2^{2j-3})) + \frac{1}{2}(1 + 2 + 8 + \ldots + 2^{2j-1}) \\ = \;& 2(1 + 2 + 8 + \ldots + 2^{2j-3}) - 2^{2j-1} \\ = \;& 2 + 4 + 16 + \ldots + 2^{2j-2} - 2^{2j-1} \leq 0, \end{aligned}$$

so Q_q cannot suffer a net increase after all pebbling out of $P_{\Delta+1} \times \{x_i\}$ is done. Considering all i and q, this means that the total value of Q can never increase, so we are done. ∎

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5 Reference.

[1] Fan R. K. Chung, Pebbling in Hypercubes, preprint.