MATH 195: Gödel, Escher, and Bach (Spring 2001)

Notes and Study Questions for Tuesday, April 3

Reading: Chapter VIII: *Typographical Number Theory* (pp.221-224 and 228-230)
A Mu Offering, pp.231-245
Chapter IX: *Mumon and Gödel*
(pp. 246-259 and pp.259-261 (OPTIONAL),
pp. 261-265 (NOT OPTIONAL)).

The reading consists of a wrap-up of Chapter 8 and an Introduction to the idea of Gödel Numbering. Although there are many pages, most of them will pose no difficulties for you, so long as you don't feel you need to understand the inner meaning of koans (if you do, then the reading will take years).

Chapter VIII: *Typographical Number Theory*

The RULE OF INDUCTION and ω - inconsistency (pp.221-224)

One of these we have discussed already: the RULE OF INDUCTION. We talked about the desirability of having an agreed upon format by which to derive theorems of the form $\forall$**a: X{a}** (infinitely many statements derived in finitely many steps). The technical aspects involved in carrying out a derivation using this rule were summarized by an outline:

- Preliminaries
- Fantasy and Generalization to establish $\forall$**a:** $<$ **X{a}** $\supset$ **X{Sa | a}** $>$
- Separate Derivation of **X{0 | a}**
- Application of the RULE OF INDUCTION.

Mike would like to add his opinion that the two most important **agreements about validity** in the history of human thought have been the RULE OF DEDUCTION (Hofstadter calls it the RULE OF DETACHMENT or MODUS PONENS, see p.186 and *Two-Part Invention*) and the RULE OF INDUCTION (p.224). The other thing to extract from pages 221-224 is the idea of **omega-inconsistency** (ω - inconsistency). The definition is on p.222.

SQ1. True/False: Supernatural numbers are numbers that correspond to NUMERALS.

SQ2. What's the difference between a number and a NUMERAL?

SQ3. Multiple Choice: If there were supernatural numbers that did not correspond to NUMERALS, then:

a) Goblins and ghouls would start to roam the earth.
b) Euclid's fifth postulate would definitely be false.
c) Euclidean geometry and **TNT** would each require a sixth axiom.
d) There might be a "pyramid" of statements, all with the same pattern **X{a}** and each one expressing a theorem of **TNT**, such that the universally quantified statement $\forall$**a:X{a}** would not be a theorem of **TNT**.

Completeness and Consistency of TNT (pp.228-230):

You may recall Hilbert's Program from Chapter I (pp.23-24). No doubt his aims were pretty mysterious back then. Perhaps they're clearer now. Number theory is tough enough even when we don't question the foundations of its logic. But that's what results when we start cracking our heads on the logical paradoxies introduced by strange loops. Hilbert hoped that a logical system could be devised (in fact HAD been devised) that could abolish strange loops and permit a systematic justification of its own internal logic. Finally, we would be able to say that a proof not only sounded right but WAS right, because the system did not permit falsehood to be produced -- it was consistent. Furthermore, if the system were also complete, a mechanistic procedure might be devised that could produce all valid theorems (similar to the procedure we devised for the **pq**-system).

SQ4. Multiple Choice: The condition most strongly related to Number Theorists going out of business is

a) the assumption that **TNT** is consistent.
b) the assumption that **TNT** is complete.
c) the assumption that **TNT** is inconsistent.
d) the assumption that **TNT** is ω – inconsistent.

SQ5. In the end, Hofstadter concludes that a valid attempt to establish the consistency of **TNT** would presumably

a) incorporate paradoxes into the system.
b) be no more difficult than establishing the consistency of the **pq**-system.
c) require the use of only the prime numbers (not the composites).
d) not require facts from number theory.

SQ6. According to Hofstadter, "circularity" is

a) round.
b) a property of numbers that are not "perfect squares".
c) inevitable.
d) morally justified.

A MU Offering

Of course, the name of this dialogue is a takeoff on *A Musical Offering*, the set of canons and other pieces given by Bach to King Frederick. The reason to substitute "MU" for "musical" will become apparent, to the extent that it CAN become apparent.

References to DNA and molecular biology run through this dialogue, some of the obvious, some of them not. This dialogue begins where the last left off, with plays on the four bases that compose DNA, Adenine, Thymine, Cytosine, and Guanine, and the first letters of the central characters: Achilles, Tortoise, Crab, and... any idea what G

could stand for? (By the way, those geometric shapes on p.236 are simplified chemical representations of the four bases)

Many references are made later to *translation*. We've done a lot of that, translating English phrases into symbolic strings, and Achilles also translates English (koans) into strings (the cellulose variety). But hidden behind this isomorphism is another, called (I'm serious) the CENTRAL DOGMA, which follows the path of genetic information:

$$\text{DNA} \xrightarrow[\textit{transcription}]{} \text{RNA} \xrightarrow[\textit{translation}]{} \text{protein}$$

The Central Dogma claims that information flows in one direction only, from DNA through messenger RNA to protein. Since our bodies are made from the action of protein, but our genes are composed of DNA, abuses to the body (like stretching the neck of a giraffe) cannot be passed on to progeny. The Central Dogma is a biochemical statement of the impossibility of Lamarckianism (that progeny inherit the environmental influences on the parents).

In fact, the Central Dogma is not quite correct. HIV is one of many viruses that use RNA as its genetic material. Its RNA is reverse transcribed to DNA for safe keeping during those long years of waiting in human cells. However, reverse translation is impossible (not merely illegal, as Achilles claims). That's because the same protein can be made from many different sets of instructions. Just as you can't recreate the instructions that produced a car by looking at the car, neither can you reverse translate protein to recreate the original RNA.

Messenger RNA is translated to protein by means of the GENETIC CODE. The string of bases that comprise messenger RNA are considered three bases at a time and translated into the amino acids that comprise a protein. For example, GAG is translated into the amino acid glutamate (also used as a flavor enhancer in Chinese restaurants). A protein typically contains hundreds of amino acids, connected like beads on a string. The order and identity of the amino acids determine how the protein will fold in three dimensions, which ultimately determines its properties.

SQ7. Describe the isomorphism between the flow of genetic information and the process by which a three-dimensional folded string is made from a koan.

Enough of DNA. What we really want to understand is the isomorphism between the manufacture of strings in TNT that correspond to truths of number theory. If we had a way of doing that that worked for all truths, we could accomplish Hilbert's aim of completeness. We need one more thing. We need some reliable test whether the resulting TNT string is or is not a theorem.

SQ8. What is the underlying meaning of Buddha nature, the String Manipulation Rules, and the Five Self-Evident Positions?

SQ9. What was the significance of Achilles tying a knot in a string that had Buddha nature?

SQ10. What was the significance of the Tortoise tying a *second* knot? Why did they both disappear?

SQ11. Starting on p.242, with the Tortoise saying, "*Now you have to realize that...*" diagram the fate of the string he produces all the way to the end of the dialogue.

SQ12. Why does Achilles say (p.244), "It's strange, to say the least," and what's bothering him?

Chapter IX: *Mumon and Gödel*

Gödel Numbering (pp.261-262):

Hofstadter has now reached the following conclusions:

- Every fact of number theory is expressible as a well-formed formula of the formal system **TNT.**
- Every known theorem of **TNT** can be interpreted as a fact of number theory.
- Every mode of reasoning employed by number theorists has been incorporated into **TNT.**

Hofstadter tells us that Gödel Numbering is a way to embed ("embed" means "to make a subset of") ***all*** problems about *any* formal system into number theory. Thus, for a particular formal system $\mathbb{F}$ we have the following sets and correspondences.

{ the set of problems about $\mathbb{F}$ } $\rightarrow$ { the set of facts of number theory }
via Gödel Numbering

{ the set of facts of number theory } $\rightarrow$ { the set of well-formed formulae of **TNT** }
via interpretation of symbols

{ the set of theorems of **TNT** } $\rightarrow$ { the set of well-formed formulae of **TNT** }
as a subset

{ the set of theorems of **TNT** } $\rightarrow$ { the set of facts of number theory }
via interpretation of symbols
*and incorporation of reasoning modes in **TNT***

SQ13. Explain each of the correspondences above.

SQ14. Can the arrows be reversed? For each arrow, explain why or why not.

SQ15. What would it mean if the third arrow could be reversed?

SQ16. What would it mean if the fourth arrow could be reversed?

SQ17. Translate each rule of MIU into a rule of 310, e.g., If $3x$ is a theorem, then ... .