Math 195: Gödel, Escher, and Bach (Spring 2001)

Exam #3

The Solutions

1. (1) If you have neither received nor given aid regarding this exam, nor have you gained or given knowledge concerning a previous or future administration of this exam, then sign your name. Otherwise sign someone else's name.

2. (1) On the first exam, many of you indicated that you thought in-class discussions in small groups were valuable for you in understanding how to do the problems. Do you still think so? What do you feel is the best use of our time in class? If you like you can also answer anonymously on the web (Discussion Board).

3. (1) The web allows us to post potentially helpful items as needed and to plan material for the next class based on what we saw in the previous class. On the other hand, many people don't avail themselves of material or discussion on the web, and there may be the cost of inconvenience by the web-based approach. Should we switch instead to a paper-based course with handouts?

4. (3) Is the string **<SS0·b ∨ ~SS0·b>** well formed? How can it be translated?

- **A.** Well-formed: "All the world's a stage"
- **B.** Well-formed: "That which we call a rose, by any other name would smell as sweet.
- **C.** Well-formed: "A horse! A horse! My kingdom for a horse!"
- **D.** None of the above (supply your own answer)

SOLUTION: There are quite a few problems with well-formedness in this string, a string that is presumably from **TNT**. There should be parentheses around **SS0·b,** there should be an equals sign somewhere. It might be possible to interpret the string as being from Propositional Calculus, with **SS0·b** somehow allowed to stand for the equivalent of **P** within that system. Still, none of the proposed translations seems to have the form of **P OR (NOT P)**, so it would be difficult to accept any answer but **D**.

With **D** chosen, your options for translation would seem to be “not well-formed, therefore uninterpretable” or perhaps the marginally less desirable “**EITHER 2*b OR NOT 2*b**”. Mike missed the clever Shakespearian “To be or not to be” translation completely!

5. (8) Translate the following into symbols to the degree possible. In the interest of partial credit, you may wish to include several translation steps, becoming more 'symbolic' on each line (a separate line for each step would be helpful to us).

- **a.** Hang gliding isn't for everyone
- **b.** To every thing there is a season, and a time to every purpose under heaven

SOLUTIONS:

a) With the variable *x* representing a person, either "It's not the case that for every person, hang gliding is for that person", i.e., **~" *x*: Hang gliding is for *x*;** or "There is a person whom hang gliding just does not suit", i.e., **$*x*: Hang gliding is not for *x*** or, if you like, **$*x*: ~Hang gliding is for *x*.** Not necessary to go further, though some may wish to replace "Hang gliding is for *x*" with something like H(x).

b) I find the universal quantifiers easiest to identify. With T as a thing and P as a purpose, we might start with "**" T: there is a season for T and " P: there is a time for P**". Continuing on to the existential quantifiers, we might let ***s*** refer to a season and let ***t*** refer to a time. Then I'd write "**" T: $s: s is for T and " P: $t: t is for P".** Finally, I can deal with the logical connective **AND,** writing "**< <" T: $s: s is for T > Ù < " P: $t: t is for P > >".**

The quantifiers can appear on the outside if you like, as long as the universal quantifiers precede the associated existential quantifiers (see problem 9), so that the following is also possible:
"" T: $s: " P: $t: < < s is for T > Ù < t is for P > >"

I chose not to associate "under heaven" with any particular word(s); it seems to me that English allows some flexibility, for example, "purpose under heaven" or "a season and a time under heaven" or perhaps every one of the nouns in the sentence is "under heaven"…

Several students moved the comma in the original statement, writing a symbolic form for "To everything, there is a season and a time for every purpose under heaven." Your English professors would say "I don't think so; you see, the author started with the parallel sentence structure `To every thing there is a season, and to every purpose there is a time.' Then, to make the sentence sound more poetic, the author inverted the second clause." The part about heaven is natural, since this verse appears in Ecclesiastes (yes, I needed to look that up).

6. (10) Derive from the propositional calculus the notion that anything follows logically if you presume a falsehood. If you like, you can derive the symbolic representation of the following: Presuming that pigs can't fly, then if pigs COULD fly, I'd be a millionaire.

SOLUTION:

An acceptable answer is found on page 196. Starting with the basic contradiction, **< P Ù ~P >**, it is possible to derive **Q** no matter what **Q** stands for. Thus "anything follows logically if you presume a falsehood."

The flying pigs version of the problem isn't very far from the text on page 196. Let **P** stand for "Pigs can fly" and let Q stand for "I am a millionaire". It may help to express what we're after in symbols: **< < ~P É < P É Q > >.** That has to be the conclusion of the derivation.

The outline of a direct derivation beginning with the “fact” that pigs can’t fly is
[~P
  [P
    …
   Q]
 < P É Q >]
< < ~P É < P É Q > >, with the … to be filled in.

Now, I see that I could get **Q** from **P** if I had **< P É Q >** to work with.
I hope you see that the detachment rule is involved in obtaining **Q** in this way.

Following p.196 more or less from line (5) to line (12), a direct derivation that works is:

(1)	**[**	push
(2)	**~P**	premise (pigs can’t fly)
(3)	**[**	push
(4)	**P**	premise (if pigs could fly)
(5)	**[**	push
(6)	**~Q**	premise (if I were not a millionaire)
(7)	**~P**	carry-over rule from (2) (pigs can’t/couldn’t fly)
(8)	**]**	pop
(9)	**< ~Q É ~P >**	fantasy rule from (6) and (7) (under the premise that pigs can’t fly, … if I were not a millionaire, then pigs couldn’t fly)
(10)	**< P É Q >**	contrapositive rule from (9) (translate the English from (9) to see that we’re essentially done)
(11)	**Q**	detachment rule from (10) and (4)
(12)	**]**	pop
(13)	**< P É Q >**	fantasy rule with (4) and (11)
(14)	**]**	pop
(15)	**< < ~P É < P É Q > >**	fantasy rule with (2) and (13).

Well, that wasn’t too bad. I see now that the derivation can be shortened to:

(1)	**[**	push
(2)	**~P**	premise
(3)	**[**	push
(4)	**~Q**	premise
(5)	**~P**	carry-over rule from (2)
(6)	**]**	pop
(7)	**< ~Q É ~P >**	fantasy rule from (4) and (5)
(8)	**< P É Q >**	contrapositive rule from (7)
(9)	**]**	pop
(10)	**< ~P É < P É Q > >**	fantasy rule from (2) and (7)

I suspected inefficiency when I saw **< P É Q >** appearing twice in my first attempt. You might note that the English translation of line (7) here agrees with line (9) above. This version of the problem was a bit different from the book's **< <PÙ~P> É Q >**, but not too much.

7. (10) A common way to prove a logical proposition is to assume that the proposition is false and show that this assumption leads to a contradiction.

a. Choose one of the strings below that you think adequately represents this strategy and show by parsing (analyzing) the string why you think so.

A. < < < P v Q > ^ ~P > É Q >
B. < < < P É Q > ^ < Q É R > > É < P É R > >
C. < < < ~P É Q > ^ < ~P É ~Q > > É P >

SOLUTION: C is correct. I translate as "if both **~P** leads to **Q** AND **~P** leads to **~Q**, then (because **< Q ^ ~Q > Û ~ < Q Ú ~Q >** would violate the Law of the Excluded Middle, pp.188-189) we may conclude that **P** must hold".

b. Construct a truth table that demonstrates that the string you chose is indeed valid.

SOLUTION: I omitted one set of angle brackets to make the truth table fit the page…

P	**Q**	**~P**	**~Q**	**~PÉQ**	**~PÉ~Q**	**<~PÉQ> ^ <~PÉ~Q >**	**< <~P É Q> ^ <~P É ~Q> > É P**
T	T	F	F	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	F	T

Truth tables for the alternatives A and B also end with a column of T's on the right.

8. (6) True or False:

a. It is possible to have a formal system without any axioms.
b. It is possible to have a formal system without any criteria for well-formedness.
c. It is possible to have a well-formed formula of TNT that does not contain the symbol "=".

SOLUTIONS: a. True, e.g. Propositional Calculus
b. False
c. False

9. (10) Say whether each statement below is true or false. If true, give some value of **c** and some value of **d** for which the statement holds. If false, then give some value of **c** and value of **d** for which the statement fails.

a. " c: $d: (SSS0 · c) = d

b. $d: " c: (SSS0 · c) = d

SOLUTIONS: **a. True, e.g., c=SS0, d=SSSSSS0. Generally, given c, set d equal to 3 times c.**

b. False, it's not true that a single d will work for every possible choice of c.

10. (8) Complete each statement and give the name of a rule illustrated by the completed statement.

Example: *If roses are red and violets are blue, then it follows that…*
Answer: *…roses are not red or violets are not blue. (via De Morgan's rule)*

a. If every good boy deserves fudge[1] and Johann is a good boy, then…

SOLUTION: … Johann deserves fudge. The specification rule is best to cite here; we go from a universal "every good boy" to a specific boy "Johann" by "substituting Johann for the variable good boy". The detachment rule seems a very good second choice; you see, it forces me to juggle the premise a bit, semi-interpreting the given statement as "If ((If Johann were a good boy then Johann would deserve fudge) and (Johann is a good boy)), then …" .

Several students gave "transitivity" as a rule illustrated by the completed statement, apparently thinking "good boy = deserves fudge AND Johann = good boy, …". This is pretty sloppy use of the symbol "=".

b. To establish that NOT all cows eat grass, it would suffice to exhibit that…

SOLUTION: There exists at least one cow that does not eat grass.
We're just switching "~" *c*: c eats grass" to "$*c*: ~ c eats grass", in accordance with the rule of interchange (p218). OK, actually in accordance with the alternate rule of interchange that comes from the stated rule via double-tilde:

$*c*: ~ c eats grass	
~~$*c*: ~ c eats grass	**double-tilde rule**
~(~$*c*:)~ c eats grass	**(my parentheses added for convenience)**
~(" *c*: ~)~ c eats grass	**interchange rule as stated (within parentheses)**
~" *c*: ~~ c eats grass	**(my parentheses removed)**
~" *c*: c eats grass	**double-tilde rule**

11. (8) Convert each of the following statements in English to an equivalent statement in English (same truth value) as indicated. Justify your answer by citing a rule of Propositional Calculus.

a. Either stones are soft OR the sky is blue. (convert to a legal IF … THEN statement)

b. Either stones are soft OR the sky is blue. (convert to a legal AND statement)

SOLUTIONS:

a. < S ∨ B > is equivalent to < ~S ⊃ B > by the Switcheroo Rule. In English,
"If stones are not soft, then the sky is blue."
b. < S ∨ B > is equivalent to ~ < ~S ∧ ~B > by DeMorgan's Rule. In English,
"The statement, 'stones are not soft and the sky is not blue' is false".
This is one of several possibilities

[1] Note that the first letters of Every Good Boy Deserves Fudge correspond to the lines of the treble clef. Therefore, no one can say that this exam is absolutely devoid of musical content.

12. (16) We wish to show that (any number) times 1 is equal to (the same number you started with). In symbols, this can be written **" c: (c×S0) = c**. The proof will be by induction. You are to help out by filling in the blanks. Note the division of the proof into the usual sections.

SECTION I: Preliminaries to derivation of < X{c} É X{Sc | c} >)

(1)	**" a:" b: (a + Sb) = S(a + b)**	Axiom 3
(2)	**" a: (a + Sc) = S(a + c)**	Specification, from line (1), replace **b** by **c**
(3)	**((c×0) + Sc) = S((c×0) + c)**	Specification, from line (2), replace **a** by **(c×0)**
(4)	**S((c×0) + c) = ((c×0) + Sc)**	Symmetry, from line (3)
(5)	**" a:" b: (a×Sb) = ((a×b) + a)**	Axiom 5
(6)	**" a: (a×S0) = ((a×0) + a)**	Specification, from line (5), replace **b** by **0**
(7)	**(c×S0) = ((c×0) + c)**	Specification, from line (6), replace **a** by **c**
(8)	**((c×0) + c) = (c×S0)**	Symmetry, from line (7), see page 219
(9)	**" a:" b: (a×Sb) = ((a×b) + a)**	Axiom 5
(10)	**" a: (a×S0) = ((a×0) + a)**	Specification, from line (9) replace b by 0, see page 217
(11)	**(Sc×S0) = ((c×0) + Sc)**	Specification, from line (10), replace **a** by **Sc**
(12)	**((c×0) + Sc) = (Sc×S0)**	Symmetry, from line (11)

SECTION II: Derivation of < X{c} É X{Sc | c} >)

(13)	**[**	Push
(14)	**(c×S0) = c**	premise (i.e., X{c})
(15)	**((c×0) + c) = (c×S0)**	Carry-over rule, from line (8)
(16)	**((c×0) + c) = c**	Transitivity, lines (14) and (15)
(17)	**S((c×0) + c) = Sc**	Add S Rule, see page 219
(18)	**S((c×0) + c) = ((c×0) + Sc)**	Carry-over rule, from line (4)
(19)	**((c×0) + Sc) = (Sc×S0)**	Carry-over rule, from line (12)
(20)	**S((c×0) + c) = (Sc×S0)**	Transitivity, page 219, from lines (18) and (19)
(21)	**(Sc×S0) = S((c×0) + c)**	Symmetry, from line (20)
(22)	**(Sc×S0) = Sc**	Transitivity, from lines (17) and (21) (i.e. X{Sc | c})
(23)	**]**	Push
(24)	**< (c×S0) = c É (Sc×S0) = Sc >**	Fantasy rule (i.e., < X{c} ⊃ X{Sc | c} >)
(25)	**" c: < (c×S0) = c É (Sc×S0) = Sc >**	Generalization, from line (24)

SECTION III: Derivation of X{0 | c}

(26)	**" a: (a×0) = 0**	Axiom 4
(27)	**(a×0) = 0**	Specification, from line (26), replace **a** with **a**
(28)	**" a:" b: (a×b) = (b×a)**	derivation omitted[2]
(29)	**" b: (a×b) = (b×a)**	Specification from line (28), replace **a** with **a**
(30)	**(a×0) = (0×a)**	Specification from line (29), replace **b with 0**
(31)	**(0·a) = (a·0)**	Symmetry, from line (30)
(32)	**(0·a) = 0**	Transitivity, from lines (31) and (27)
(33)	**" a: (0×a) = 0**	Generalization, from line (32)
(34)	**(0×S0) = 0**	Specification, from line (33), replace **a** by **S0** (i.e., X{0 | c})

SECTION V: Bottom line

(35)	**" c: (c×S0) = c**	Induction, from lines (25) and (34)

[2] We would need many tens more lines here (similar to long derivation on pp.225-227)