Exam 2 Solutions

The suggestions and comments you wrote in questions 2 and 10can be found on the Discussion Board.

5. (2) Kurt Gödel proved that sufficiently strong formal systems must contain well-formed strings that cannot be classified as either theorems or nontheorems. This result has become known as (choose one):

A. Gödel's Incompleteness Theorem

B. Gödel's Inconsistent Theorem

C. Gödel's Incomprehensible Theorem

D. Gödel's Incom- (at this point the author died)

SOLUTION: **A**. Gödel's Incompleteness Theorem

6. (8) Consider the **pq**-system with the following additional axiom schema:

AXIOM SCHEMA II: If ***x*** is a hypen-string, then ***x*** **p – q** ***x*** **- -** is an axiom

6a. From the two axiom schema (the original and II) derive the theorem:

- - - p - - - q - - - - - - - [7 hyphens]

SOLUTION: a) Realize that the system has only one RULE; work backwards to see that
- - **p** - - - **q** - - - - - - - comes from - - - **p** - - **q** - - - - - -,
which in turn comes from - - - **p** - **q** - - - - -, which is an axiom.

The derivation should be written forward.

- - - **p** - **q** - - - - - (axiom)
- - - **p** - - **q** - - - - - - (apply rule)
- - - **p** - - - **q** - - - - - - - (apply rule again)
DONE.

6b. Now add a third axiom schema:

AXIOM SCHEMA III: If ***x*** is a hypen-string, then ***x*** **p – q** ***x*** is an axiom

Think of an interpretation to make the new system (with three axiom schema) consistent with the external world. *[Note: this problem is by no means easy. Go as far with it as you can, but don't worry too much if you can't come up with a satisfying answer.]*

SOLUTION: b) With all three axioms, the truths corresponding to well-formed strings *x* **p** *y* **q** *z* might be expressed as "*x* plus *y* differs from *z* by at most one unit". Such a system might be useful in studying the consequences of "off-by-one" addition errors.

Some of you thought that the system could be interpreted as "*x* plus *y* is less than or equal to or greater than *z*". This interpretation is not completely satisfying, even though every theorem in the new system corresponds to a true statement within the interpretation. The reverse, however, is not true. There are an infinite number of true statements in the new system that do not correspond to theorems. Examples include 1+1 < 5. **– p – q - - - - -** is not a theorem in the new system.

7. (10) Each theorem of the *Exam2-system* is a positive integer. The system is described by:

10 SYMBOLS: **0 1 2 3 4 5 6 7 8 9**

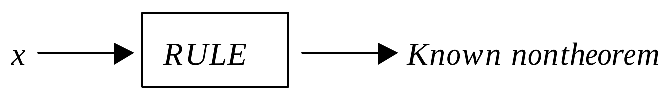
2 AXIOMS:

AXIOM 1: **10** is a theorem
AXIOM 2: **11** is a theorem

2 RULES (to be interpreted arithmetically):
RULE 1: If ***x*** is a theorem of the system, then ***x***+***x*** is also a theorem.
RULE 2: If ***x*** and ***y*** are theorems of the system, then ***x***+***y*** is also a theorem.

7a. Given that **36** is a nontheorem of *Exam2-system with **10** and **11** as axioms*, work backward to generate at least 5 more nontheorems of the system.

HINT: Consider that you are looking for nontheorems (call them ***x***) that satisfy the following condition:



If you find such an x, you KNOW that it must be a nontheorem, because if it were a THEOREM, application of the rule must lead to another theorem.

SOLUTION: a) Since 36 is a nontheorem while 10 and 11 are theorems, you know that 18, 26, and 25 are nontheorems. I get these three numbers by working backward, seeing that 36 would have come from 18 via Rule 1, from 26 and 10 via Rule 2, and from 25 and 11 via Rule 2 also.
Since 18 is a nontheorem, so are 9, 8, and 7, by same working backward process.
Since 26 is a nontheorem, so are 13, 16, and 15.
Since 25 is a nontheorem, so are 15 (again) and 14.
I guess you can go one more level up from 36 and conclude (from 16, 15, 14, and 13) that 6, 5, 4, 3, and 2 are nontheorems. Finally, from 2 you can get that 1 is a nontheorem.

There was another approach used by a student who employed only Backwards Rule 1 to get from 36 to 18 and 9 as above; then [since ((12+12)+12)=36], the student concluded that 12, 6, and 3 were nontheorems as well… I won't take this any further.

Many of you thought that (36+36), (36+36+36), etc must also be nontheorems. This isn't necessarily the case. The idea rests on a misuse of Rule 1. Rule 1 says that if ***x*** is a theorem, so is ***x*** + ***x***, but the rule says nothing about the consequences of ***x*** NOT being a theorem. For example. 5 is certainly not a theorem, but 5+5 is.

7b. Formalizing the working backward process for the *Exam2-system with **10** and **11** as axioms* is a bit different from what we did in class with MIU. Do your best to formulate ONE working backward rule by completing the following:

RULE FOR *Backward Exam2-system*:
If ***z*** is a nontheorem of the original *Exam2-system with **10** and **11** as axioms*, then _____ is also a nontheorem.

SOLUTION: b) My three candidates for *Backward Exam2-system Rules* were

If z is a nontheorem of the *Exam2-system with 10 and 11 as axioms*,
then (provided z is even) $z/2$ is also a nontheorem.
If z is a nontheorem of the *Exam2-system with 10 and 11 as axioms*,
then $z - 10$ is also a nontheorem.
If z is a nontheorem of the *Exam2-system with 10 and 11 as axioms*,
then $z - 11$ is also a nontheorem.

The student mentioned above might have proposed this:
If z is a nontheorem of the *Exam2-system with 10 and 11 as axioms*,
then (provided k is a divisor of z) z/k is also a nontheorem.

8. (16) Consider the term "theorem schema"

8a. What could this term mean? Give as precise a definition as you can, but try to achieve enough generality so that your definition can be applied in a variety of different formal systems.

SOLUTION: a) The term "theorem schema" should mean "a symbol string containing variables with the property that (IF ((the same well-formed string is substituted for each instance of each variable) AND (the resulting symbol string is well-formed)) THEN the resulting well-formed string is a theorem of the system)". I was very careful with well-formedness of the resulting string. In more plain English, a "theorem schema" should "capture" many theorems of a system through the use of variables.

Many students consulted the text (p47) and wrote "a mold in which all theorems are cast". This cost each student ½ a point since it is fairly clear that the example used in the rest of the problem does not capture "all" the theorems of the **tq**-system.

8b. In the **tq**-system, the string **- - t *x* q *xx*** might qualify as a theorem schema. Explain.

SOLUTION b) You want to explain "every string of this form is a theorem" and not "every theorem has this form". The string - - **t** *x* **q** *x x* may be interpreted as "2 times *x* equals (*x* + *x*)". If we assume that the **tq**-system is a complete model for multiplication with two factors, then the algebraic truth of the interpreted statement implies theoremhood for the symbol string within **tq**. You should think about why it is completeness and not consistency that matters here.

You may also choose not to rely on an interpretation. Focussing only on the axioms and rules of the tq-system might lead you to the following observations:

- - **t** - **q** - - is an axiom (*x* is "-")
- - **t** - - **q** - - - - is a theorem obtained by applying the rule to the axiom above (*x* is "- -")
- - **t** - - - **q** - - - - - - is a theorem obtained by applying the rule to the axiom above (*x* is "- - -")
etc.

From these observations it is natural to conclude that - - t *x* q *x x* is a theorem whenever *x* is replaced by a finite string of hyphens.

8c. Which axiom(s) of the **tq**-system would be involved in deriving the theorem:

- - t - - - q - - - - - -

Explain why.

SOLUTION c) Working backward from - - **t** - - - **q** - - - - - - via the one Rule of production gives - - **t** - - **q** - - - -, and then - - **t** - **q** - - . This is the relevant axiom.

8d. Which axiom(s) of the **tq**-system would be involved in deriving the theorem:

- - t - - - - q - - - - - - - -

Generalize to determine which axiom(s) of the tq-system would be involved in deriving theorems that fit the theorem schema **- - t *x* q *xx***.

SOLUTION d) Working backward from the given theorem via the one Rule of production gives - - **t** - **q** - - as the relevant axiom just as in part c). Generalizing, it seems that - - **t** - **q** - - is the axiom from which all theorems captured by the schema - - **t** *x* **q** *x x* are derived.

Visually, you might see that the proposed theorem schema captures all theorems in the column of the generating tree for **tq** whose first entry is - - **t** - **q** - - .

8e. In the next sections of the book, we'll work to develop criteria by which one might PROVE that **- - t *x* q *x x*** is indeed a theorem schema. Based on your work in **8a-d**, do your best to express a convincing proof according to what YOU would require of such a proof. Any clear and thoughtful response will receive some credit. (Suggestion: Use few words per line)

SOLUTION e) Here's a fancy response that boils down to something substantive.

The schema becomes a theorem when "-" is substituted for *x*.

If the schema becomes a theorem when a particular hyphen string is substituted for *x*, then, by applying the rule, we see that the schema also becomes a theorem when a hyphen string of length one more than the particular string is substituted for *x*.
(Note: recognizing that *x* - *x* - is the same as *x x* - - on the right hand side is important here.)

It is useful to think of the first statement above as an axiom and to think of the second statement as a rule of production. Then you see that this axiom and rule produce theorems in the form of the schema when ANY finite hyphen string is substituted for *x*; just work backwards to the axiom.

Another way of looking at it:

1. Suppose that **- - t *x* q *x x*** is a theorem	Premise
2. Then **- - t *x* - q *x x* - -** is also a theorem	Rule
3. Equivalent to **- - t *x* – q *x* - *x* -**	Rearrangement

*Note that **x** represents a string of hyphens. I'm free to mentally group hyphens as I choose.*

4. Equivalent to **- - t *y* q *y y***	Let ***y*** = ***x*** **–**
5. Equivalent to **- - t *x* q *x x***	Variable name

Note that I'm free to name variables as I choose, so long as the naming is consistent within the string.

6. So, if **- - t *x* q *x x*** is a theorem, so is the next one in the series (with ***x***+1 hyphens), and the next, and the next.
7. It remains to be shown that the FIRST string of the form **- - t *x* q *x x*** is a theorem
8. **- - t - q - -** is a theorem — Axiom
9. Therefore ALL strings of the form are theorems.

9. (16) Take a sheet of blank paper (you can do this mentally if you like). Call it the **Input Rectangle**. Determine the blank area of the paper. . . actually, don't bother: let's just call it 1 unit.

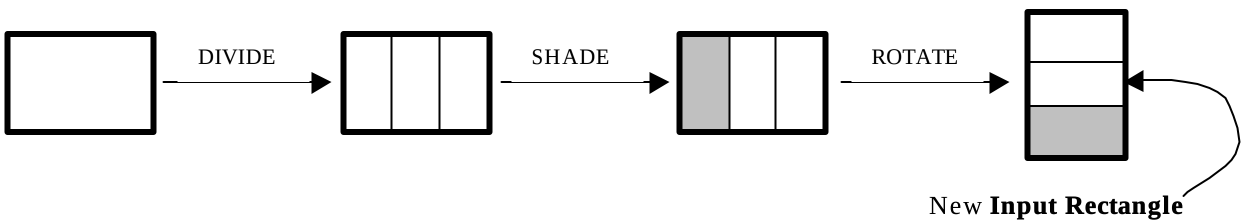
TotalBlankArea = 1

Determine the shaded area of the paper. There isn't any:

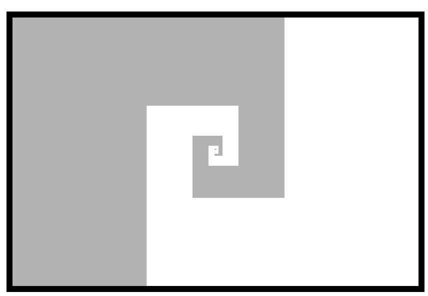
TotalShadedArea = 0

Now, perform on the **Input Rectangle** the following operations (see figure on next page):

- DIVIDE the **Input Rectangle** in thirds
- SHADE the left-most third of the **Input Rectangle**
- ROTATE the **Input Rectangle** one-quarter turn counter-clockwise
- ADD the area you just shaded to the **TotalShadedArea**
- SUBTRACT the area you just shaded from the **TotalBlankArea**
- COMPARE the **TotalShadedArea** with the **TotalUnshadedArea**
  - If the two areas are equal then STOP
  - If they are not equal then repeat the instructions DIVIDE through COMPARE, using the (middle) rectangle above the newly shaded rectangle as the new **Input Rectangle**
  -



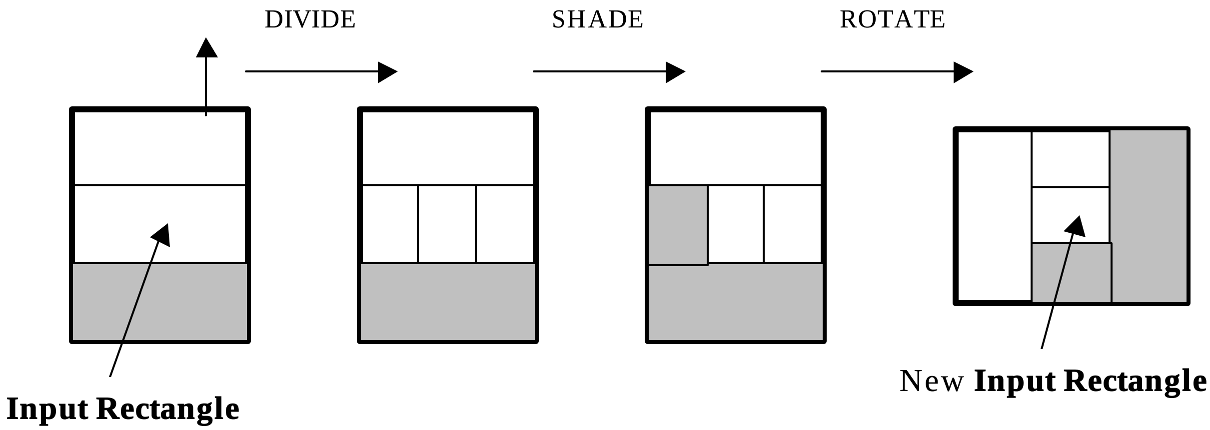
Here's what the procedure gives after several iterations:



9a. Using the same format as shown in the example on the previous page, continue the process through the next round, drawing each step and the process that produced it.

SOLUTION

Between the ROTATE step and the COMPARE step in the iteration shown on the Exam, the values of **TotalBlankArea** and **TotalShadedArea** are reset so that the value of **TotalBlankArea** = 2/3 and the value of **TotalShadedArea** = 1/3 when we start the second iteration.



Between the ROTATE step and the COMPARE step in the iteration shown here, the values of **TotalBlankArea** and **TotalShadedArea** are reset so that value of **TotalBlankArea** = 2/3 – 1/9 = 5/9 and value of **TotalShadedArea** = 1/3 + 1/9 = 4/9. Since the two are not equal, we identify a new input rectangle.

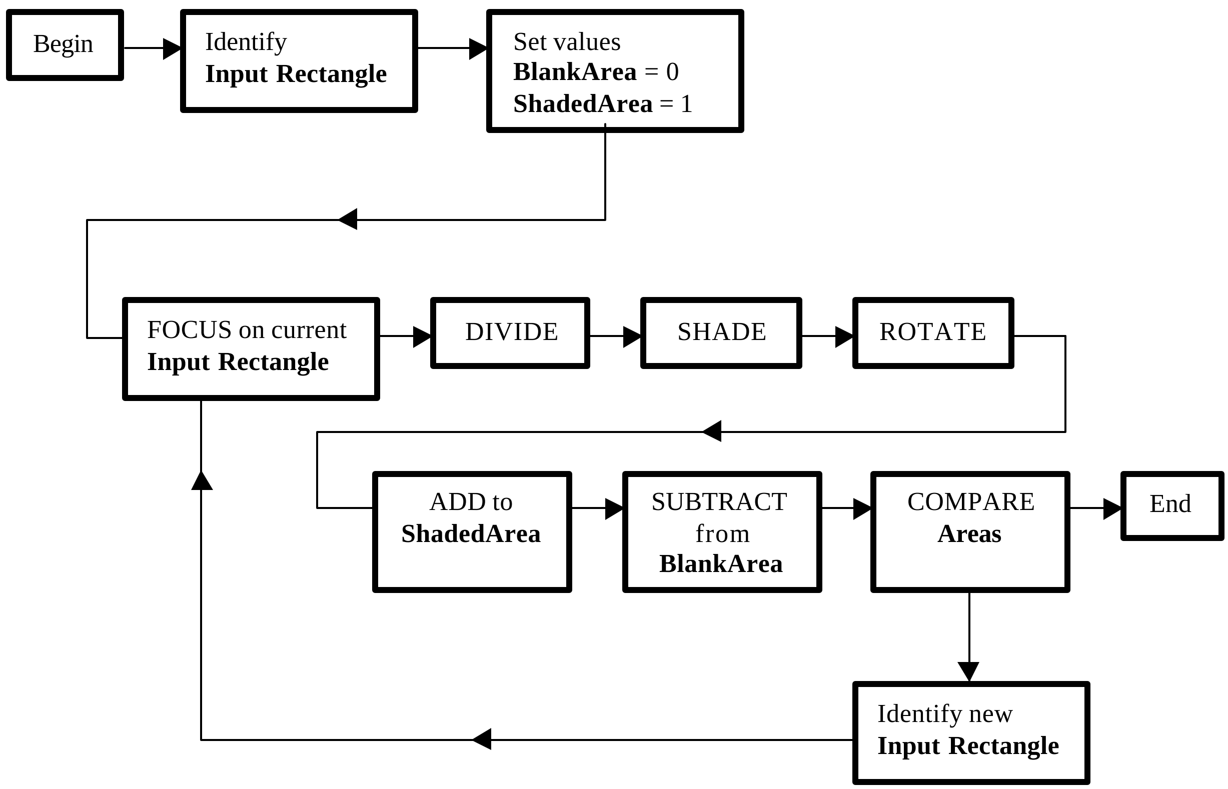
Many of you extracted the input rectangle from out of the larger rectangle and ran it through the instructions in isolation. This procedure is plausible but it wouldn't give pattern shown next to "Here's what the procedure gives after several iterations:"

9b. Use Figures 27 and / or 31 to help you draw a Recursive Transition Network for the process. Be sure to include nodes labeled "BEGIN" and "END" and to identify the input to the RTN.

SOLUTION

A suitable RTN is shown below. There are two important features to notice:

1) There is an INITIALIZATION consisting of the instructions on the top line.
2) There is only one decision made; the decision to STOP or CONTINUE.



9c. Will this process ever terminate? Explain your answer.

SOLUTION c) NO. At each stage of the process, the shaded area is less than ½. The only way the process could terminate in finite time (still infinitely many iterations) would be if each successive stage could be executed "twice as fast"….

9d. The big expanded rectangle above appeared in this month's issue of *The College Mathematics Journal* with the caption:

$$\mathbf{1/3 + (1/3)^2 + (1/3)^3 + \ldots = 1/2}$$

Do your best to explain this formula in terms of areas:

The right side of the formula represents _____________.

The left side of the formula represents ______________.

SOLUTION d) The left side of the formula represents the total shaded area obtained via "shading one third of the input rectangle, then repeating the process on the 'next' third". The right side of the formula represents the 'eventual' big picture wherein ½ of the area is shaded and ½ remains unshaded.

Some of you thought that the left hand side represented the shaded area and the right hand side the unshaded area. Others thought just the reverse. There's no reason to equate the right hand side with either possibility.

9e. Explain what convinces you that the caption is appropriate. You may argue in terms of the rectangle or, if you prefer, you may make an algebraic argument.

SOLUTION e) I just did in 9d). As the process continues, the shaded area grows toward ½ while the unshaded area shrinks to ½ . Note that the shapes of the shaded and unshaded regions appear to be identical (rotate the unshaded area 180°). If they're identical, then their areas must be the same, so each must represent half the area of the whole.

We wanted to leave the possibility of an algebraic solution available to students. Algebra produces the caption as follows. Write the sum as

$$S = 1/3 + (1/3)^2 + (1/3)^3 + \ldots$$

Then $$3S = 1 + 1/3 + (1/3)^2 + (1/3)^3 + \ldots$$

$$= 1 + S,$$

subtract to get $2S = 1$

so that $S = ½$.