Math 195: Gödel, Escher, and Bach (Spring 2001)

Welcomes you to Exam #2

----Same as before -----

RULES OF THE GAME: This is an open book exam. It is an open notes exam. Needless to say, it is not an open people exam. This applies particularly to communication between people taking the exam at different times.

ANSWER SHEET: Put answers any paper you like, but NOT on the exam itself (it would just be a mess). Don't bother rewriting the question on your answer sheet, just the answers themselves – as neatly and logically as you can. The exam is yours to keep.

THOUGHT SHEET: Reserve your answer sheet for answers. However, you are encouraged to write on a separate sheet (labeled "thoughts") the thought processes that led you to those answers or, if you're stuck, what sticks you.

WEIGHTS OF QUESTIONS: See the numbers in the parentheses for each question.

NEED A CLUE? If you've exhausted your own resources with respect to one of the questions, consider coming to Mike or Jeff for a hint to get you back on track. Some hints may cost a fraction of the total points you receive for the question, but that may be better than getting nothing at all. You'll be advised of any charge before receiving a hint.

The Questions

1. (1) If you have neither received nor given aid regarding this exam, nor have you gained or given knowledge concerning a previous or future administration of this exam, then sign your name. Otherwise sign someone else's name.

2. (2) Consider the Discussion Board, which you can get to by clicking on Communication.

2a. Did you know that the Discussion Board has a forum where you can (anonymously if you like) set forth your complaints and suggestions as to how the course could be improved?

2b. One suggestion made was that exams should be held in class. This would mean that exams would be strictly timed (necessary, since some people have obligations immediately after class). It would also cut into the time available to us to discuss problems, presumably making it more difficult for you to do those problems. On the other hand, it would make scheduling easier. What do you think?

3. (5) How many quizzes did you take of the five made available? (If you don't remember, don't worry about it)

4. (10) Did you hand in the assigned homework taken from Problem Sets 5 and 6?

5. (2) Kurt Gödel proved that sufficiently strong formal systems must contain well-formed strings that cannot be classified as either theorems or nontheorems. This result has become known as (choose one):

A. Gödel's Incompleteness Theorem

B. Gödel's Inconsistent Theorem

C. Gödel's Incomprehensible Theorem

D. Gödel's Incom- (at this point the author died)

6. (8) Consider the **pq**-system with the following additional axiom schema:

AXIOM SCHEMA II: If ***x*** is a hypen-string, then ***x*** **p – q** ***x*** **- -** is an axiom

6a. From the two axiom schema (the original and II) derive the theorem:

- - - p - - - q - - - - - - - [7 hyphens]

6b. Now add a third axiom schema:

AXIOM SCHEMA III: If ***x*** is a hypen-string, then ***x*** **p – q** ***x*** is an axiom

Think of an interpretation to make the new system (with three axiom schema) consistent with the external world. *[Note: this problem is by no means easy. Go as far with it as you can, but don't worry too much if you can't come up with a satisfying answer.]*

7. (10) Each theorem of the *Exam2-system* is a positive integer. The system is described by:

10 SYMBOLS: **0 1 2 3 4 5 6 7 8 9**

2 AXIOMS:

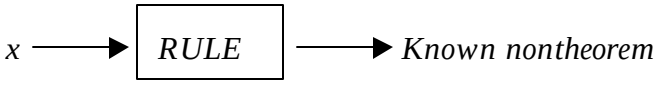
AXIOM 1: **10** is a theorem
AXIOM 2: **11** is a theorem

2 RULES (to be interpreted arithmetically):

RULE 1: If ***x*** is a theorem of the system, then ***x***+***x*** is also a theorem.
RULE 2: If ***x*** and ***y*** are theorems of the system, then ***x***+***y*** is also a theorem.

7a. Given that **36** is a nontheorem of *Exam2-system with **10** and **11** as axioms*, work backward to generate at least 5 more nontheorems of the system.

HINT: Consider that you are looking for nontheorems (call them ***x***) that satisfy the following condition:



If you find such an x, you KNOW that it must be a nontheorem, because if it were a THEOREM, application of the rule must lead to another theorem.

7b. Formalizing the working backward process for the *Exam2-system with **10** and **11** as axioms* is a bit different from what we did in class with MIU. Do your best to formulate ONE working backward rule by completing the following:

RULE FOR *Backward Exam2-system*:

If ***z*** is a nontheorem of the original *Exam2-system with **10** and **11** as axioms*, then _____ is also a nontheorem.

8. (16) Consider the term "theorem schema"

8a. What could this term mean? Give as precise a definition as you can, but try to achieve enough generality so that your definition can be applied in a variety of different formal systems.

8b. In the **tq**-system, the string **- - t *x* q *xx*** might qualify as a theorem schema. Explain.

8c. Which axiom(s) of the **tq**-system would be involved in deriving the theorem:

- - t - - - q - - - - - -

Explain why.

8d. Which axiom(s) of the **tq**-system would be involved in deriving the theorem:

- - t - - - - q - - - - - - - -

Generalize to determine which axiom(s) of the tq-system would be involved in deriving theorems that fit the theorem schema **- - t *x* q *xx***.

8e. In the next sections of the book, we'll work to develop criteria by which one might PROVE that **- - t *x* q *x x*** is indeed a theorem schema. Based on your work in **8a-d**, do your best to express a convincing proof according to what YOU would require of such a proof. Any clear and thoughtful response will receive some credit. (Suggestion: Use few words per line)

9. (16) Take a sheet of blank paper (you can do this mentally if you like). Call it the **Input Rectangle**. Determine the blank area of the paper. . . actually, don't bother: let's just call it 1 unit.

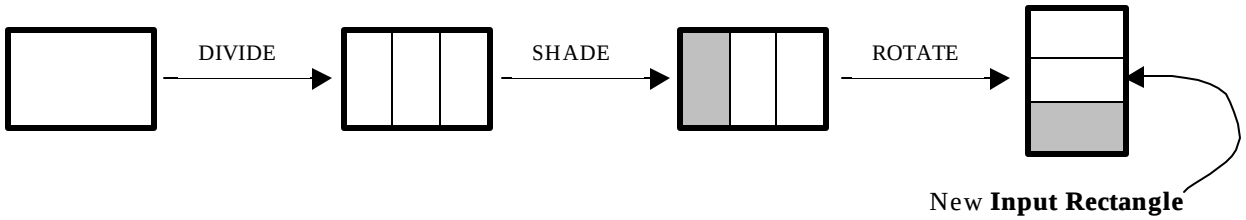
TotalBlankArea = 1

Determine the shaded area of the paper. There isn't any:

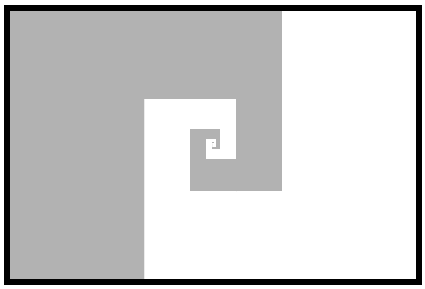
TotalShadedArea = 0

Now, perform on the **Input Rectangle** the following operations (see figure on next page):

- DIVIDE the **Input Rectangle** in thirds
- SHADE the left-most third of the **Input Rectangle**
- ROTATE the **Input Rectangle** one-quarter turn counter-clockwise
- ADD the area you just shaded to the **TotalShadedArea**
- SUBTRACT the area you just shaded from the **TotalBlankArea**
- COMPARE the **TotalShadedArea** with the **TotalUnshadedArea**
    - If the two areas are equal then STOP
    - If they are not equal then repeat the instructions DIVIDE through COMPARE, using the (middle) rectangle <u>above</u> the newly shaded rectangle as the new **Input Rectangle**



Here's what the procedure gives after several iterations:



9a. Using the same format as shown in the example on the previous page, continue the process through the next round, drawing each step and the process that produced it.

9b. Use Figures 27 and/or 31 to help you draw a Recursive Transition Network for the process. Be sure to include nodes labeled "BEGIN" and "END" and to identify the input to the RTN.

9c. Will this process ever terminate? Explain your answer.

9d. The big expanded rectangle above appeared in this month's issue of *The College Mathematics Journal* with the caption:

$$\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots = \frac{1}{2}$$

Do your best to explain this formula in terms of areas:

The right side of the formula represents _____.

The left side of the formula represents _____.

9e. Explain what convinces you that the caption is appropriate. You may argue in terms of the rectangle or, if you prefer, you may make an algebraic argument.[1]

10. (2) What are your thoughts upon completing this exam? (*Examples: satisfaction, suicidal longings, nothingness,. . .*)

[1] For the sake of those around you, please muffle any hysterical laughter.