Psychological Bulletin
1966, Vol. 66, No. 4, 252–262

A GENERAL EQUATION AND TECHNIQUE FOR THE EXACT PARTITIONING OF CHI-SQUARE CONTINGENCY TABLES[1]

JEAN L. BRESNAHAN AND MARTIN M. SHAPIRO
Emory University

This paper considers the technique of the exact partitioning of χ^2 contingency tables. Methods are presented for partitioning contingency tables into components. A general equation for χ^2 is derived. The equation may be used for the calculation of exact χ^2 values for (a) nonexhaustive sets of categories, and (b) situations in which some cells have small expected frequencies.

Brief references to the partitioning of chi-square contingency tables have appeared in the statistical literature for many years (e.g., Fisher, 1932), but few detailed discussions of the topic exist. Essentially, until this time there has been a formula (Irwin, 1949) for partitioning into 2×2 tables, its computational equivalent (Kimball, 1954), and an equation (Kastenbaum, 1960) for applications in which one dimension has been reduced and the remaining table has more than one degree of freedom. In a source more readily available to psychologists, Castellan (1965) has recently reviewed some of these procedures.

This paper will present additional, more complex patterns of partitioning, supplemented by a general equation not heretofore formulated. Furthermore, it can be shown that this general equation is applicable for any mode or degree of reduction of a contingency table and that the technique provides a most suitable solution to the problem of nonexhaustive categories or expected frequencies which are too small.

Purpose of Partitioning

When a chi-square test of independence is calculated for a contingency table of more than one degree of freedom by the usual formula

$$\chi_{(r-1)(c-1)}^2 = \sum \frac{(f_{oij} - f_{eij})^2}{f_{eij}} \qquad [1]$$

where r = number of rows, c = number of columns, f_{oij} = observed frequency in cell ij ($i = 1, \cdots, r$; $j = 1, \cdots, c$), f_{eij} = expected frequency in cell ij ($i = 1, \cdots, r; j = 1, \cdots, c$), statistical interpretation of the outcome is difficult. A significant value of chi-square indicates nonindependence of the variables, but provides no information regarding whether nonindependence occurs throughout the whole table, or in any specific part of the table. A nonsignificant value indicates that for the table considered as a whole there is independence, but provides no information regarding the possibility of nonindependence within specific parts of the table.

By partitioning the total chi-square it is possible to make additional comparisons of cells within the whole table. It has been shown (Irwin, 1949; Kimball, 1954; Lancaster, 1949, 1950; Maxwell, 1961) that an $r \times c$ contingency table with $(r - 1)(c - 1)$ degrees of freedom can be subdivided into $(r - 1)(c - 1)$ 2×2 tables, each with one degree of freedom. The chi-squares of the individual partitions are classified as additive components, their sum being equal to the chi-square calculated for the whole table. Castellan (1965) has provided a detailed summary exposition of these procedures and those for partitioning $2 \times N$ tables into components of more than one degree of freedom.

For computing chi-square on partitioned tables of one or more degrees of freedom, Kastenbaum (1960) has provided a formula which is applicable to situations where only one dimension has been reduced, that is, either rows or columns, but not both:

$$\chi_{(r-1)(m-1)}^2 = N \sum_{i=1}^{r} \frac{1}{n_{i.}} \left[\sum_{j=1}^{m} \frac{n_{ij}^2}{n_{.j}} - \frac{(\sum_{j=1}^{m} n_{ij})^2}{\sum_{j=1}^{m} n_{.j}} \right] \qquad [2]$$

[1] This research was supported in part by grants HD-00927 and HD-01333 from the National Institutes of Health, United States Public Health Service.

where r = number of rows; c = number of columns in the whole table; m = number of columns in the partitioned table, $m < c$; n_{ij} = observed frequency in cell ij;

$$n_{.j} = \sum_{i=1}^{r} n_{ij}$$

$$n_{i.} = \sum_{j=1}^{c} n_{ij}$$

$$N = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij} = \sum_{j=1}^{c} n_{.j} = \sum_{i=1}^{r} n_{i.}$$

This procedure has greater generality than those reviewed by Castellan (1965) because it is not restricted to partitions of one degree of freedom from $r \times c$ tables or to partitions of more than one degree of freedom from $2 \times N$ tables. The equation may be used to determine the contribution of a portion of the table to the overall chi-square, but with the requirement that only the number of rows or only the number of columns has been reduced.

Determination of Permissible Partitions

Irwin (1949), Kimball (1954), Lancaster (1949, 1950), and Maxwell (1961) have discussed a procedure for determining the permissible 2×2 partitions. The scheme entails beginning in one corner of the table and systematically isolating and eliminating the single element appearing in that corner. For every row and column, this procedure results in comparisons between the first cell and all succeeding cells pooled, between the second cell and all succeeding cells pooled, and so on, until the final comparison between the second-last and last cells. An example of partitions of this kind for a 4×4 table is shown in Figure 1.

It is easy to demonstrate that other schemes for partitioning exist, and Kastenbaum (1960) has given an example; but if the conventional plan is analyzed, three general rules are apparent: (*a*) Each element appears by itself once and only once. (*b*) The same combination of elements do not appear more than once. (*c*) The dividing lines of the partitions are invariant in that once used, no elements may be combined across them in future partitions.

By following these three rules it is possible to create new partitioning schemes, thus further increasing the number of possible sets of partitions and comparisons for any given table. An example of such a different kind of scheme is illustrated in Figure 2 for a 4×4 table.

Calculation of Chi-Square for Partitions

Irwin (1949) has presented a general formula for the $(r-1)(c-1)$ independent 2×2 tables partitioned from the whole table:

$$\chi_1^2 = \frac{E}{e_{1.}e_{.1}e_{2.}e_{.2}} \times (a_{11}e_{22} + a_{22}e_{11} - a_{12}e_{21} - a_{21}e_{12})^2 \quad [3]$$

where a_{ij} = observed frequency in cell ij of the 2×2 table $(i, j = 1, 2)$; $e_{i.}$ = the sum of the expected frequencies calculated from the margins of the original table for cells $i1$ plus $i2$ $(i = 1, 2)$; $e_{.j}$ = the sum of the expected frequencies calculated from the margins of the original table for cells $1j$ plus $2j$ $(j = 1, 2)$; and

$$E = \sum_{i=1,2} e_{i.} = \sum_{j=1,2} e_{.j}$$

This formula reduces to the usual expression for a 2×2 table if the expected frequencies are calculated from the margins of the 2×2 table itself; and not, as here, from the margins of the original table.

To illustrate the use of this formula, the 4×4 table discussed previously (Figs. 1 and 2) is now presented with actual numbers and chi-square values calculated for each of the partitions (Fig. 3). For both methods of partitioning, the sum of the χ_1^2 is equal to the total χ_9^2.

Snedecor (1956) has shown that the formula

$$\chi_{(N-1)}^2 = \frac{(\sum x_j p_j) - (\hat{p}T_x)}{\hat{p}\hat{q}} \quad [4]$$

where n_j = margin frequency in category j; $j = 1, \cdots, N$; $n_j = x_j + x'_j$; x_j = observed frequency in one cell of category j; and

$$p_j = \frac{x_j}{n_j}$$

$$T = \sum_{j=1}^{N} x_j + \sum_{j=1}^{N} x'_j = \sum_{j=1}^{N} n_j$$

$$T_x = \sum_{j=1}^{N} x_j$$

$$\hat{p} = T_x/T$$

$$\hat{q} = 1 - \hat{p}$$

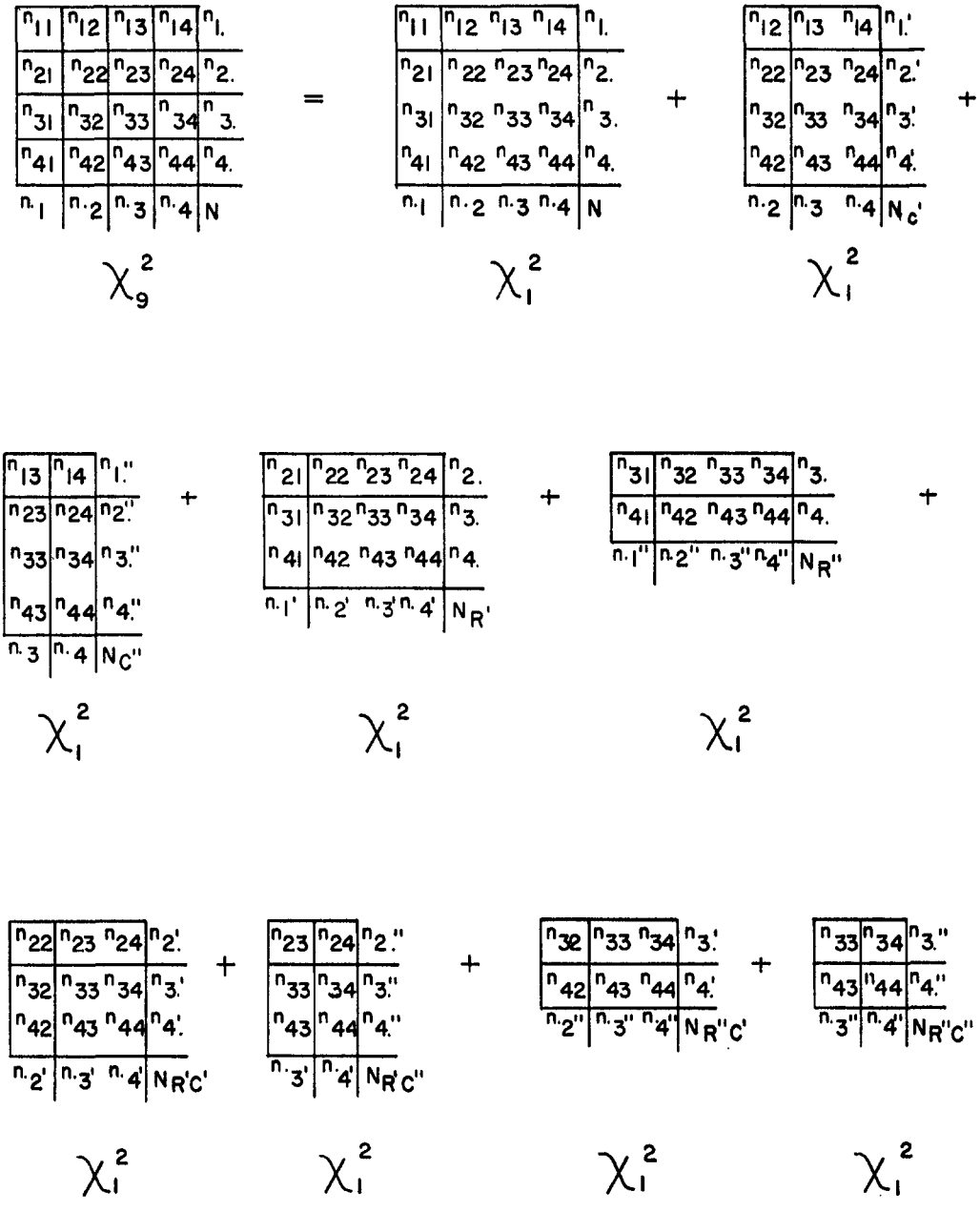

FIG. 1. Example of partitioning in a 4 × 4 table using the conventional method of successively eliminating one column and one row.

can be used for the total chi-square of a $2 \times N$ contingency table. An equivalent formula for a $2 \times N$ table using frequencies can be found in Walker and Lev (1953). Cochran (1954) further noted that when the denominator is kept constant, the formula is equal to χ_1^2 for a 2×2 partition from a $2 \times N$ table. If this new expression is converted from proportions to frequencies, the following formula results:

$$\chi^2 = \sum \frac{(f_o - f_e)^2}{F_e} \qquad [5]$$

where f_o = observed frequency of partitioned table; f_e = expected frequency of partitioned table calculated from marginal totals of partitioned table; and F_e = expected frequency of partitioned table calculated from marginal totals of original table.

It can be demonstrated that this formula is applicable to 2×2 partitions, not only from a $2 \times N$ table, but from any $r \times c$ table, the sole restriction being that only one dimension has been reduced. That is, chi-square calculated by this formula equals chi-square calculated by Irwin's formula if one or more rows (or columns) have been sliced off, provided that all columns (or rows) remain. The first five partitions for the 4×4 table meet the criterion of either complete rows or columns, and the chi-square values for these partitions can be obtained by either formula.

Equation 5 is also equivalent to Equation 2 (the proof requires only algebraic manipulations) since for any partition which includes all the categories of one dimension, the sum of the expected frequencies as determined by the original marginal totals is equal to the sum of the observed frequencies. An alternative statement is that the sum of the expectations is equal to the expectation of the sum; and, an even stronger statement is that the sum of the original expectations is equal to the sum of the observed in either each row or each column (but not both). This relationship does

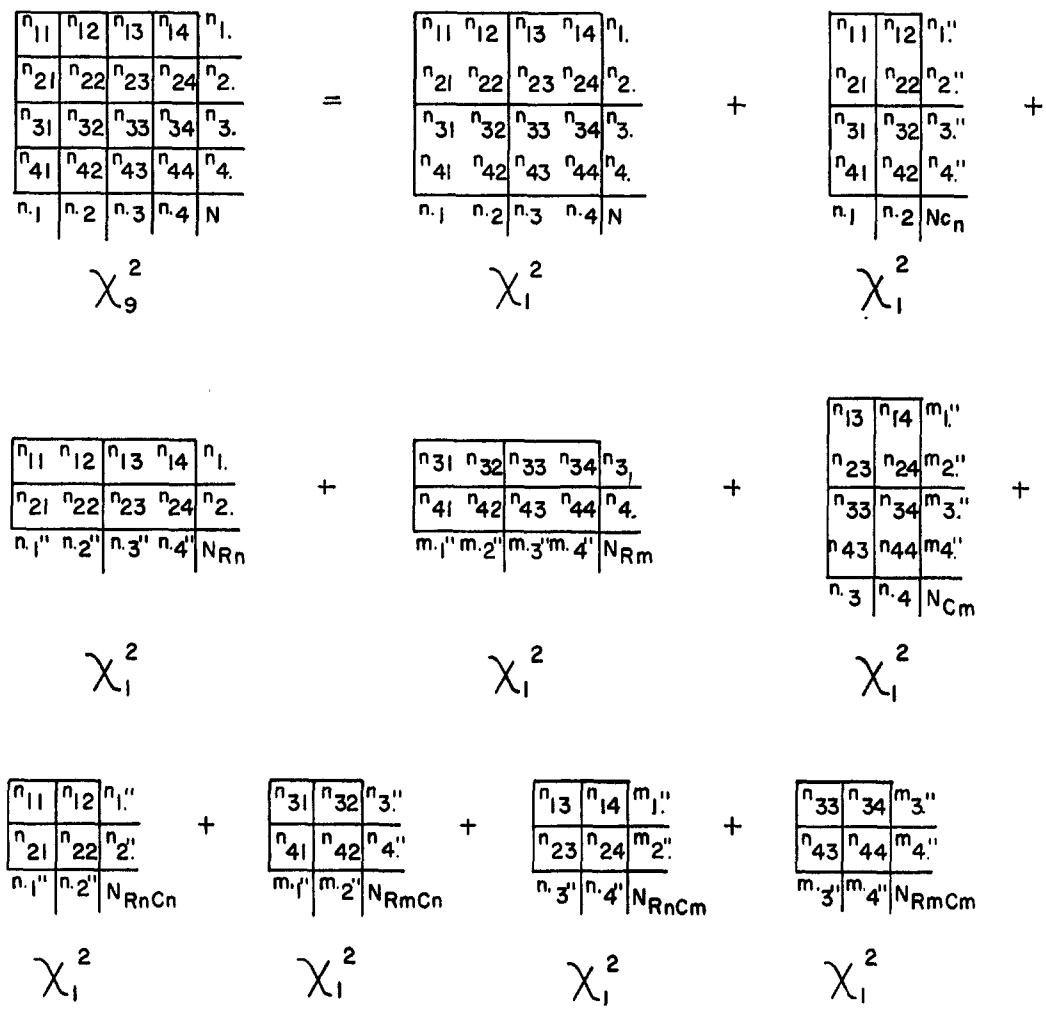

FIG. 2. Example of a partitioned 4×4 table using a variant of the conventional method of partitioning.

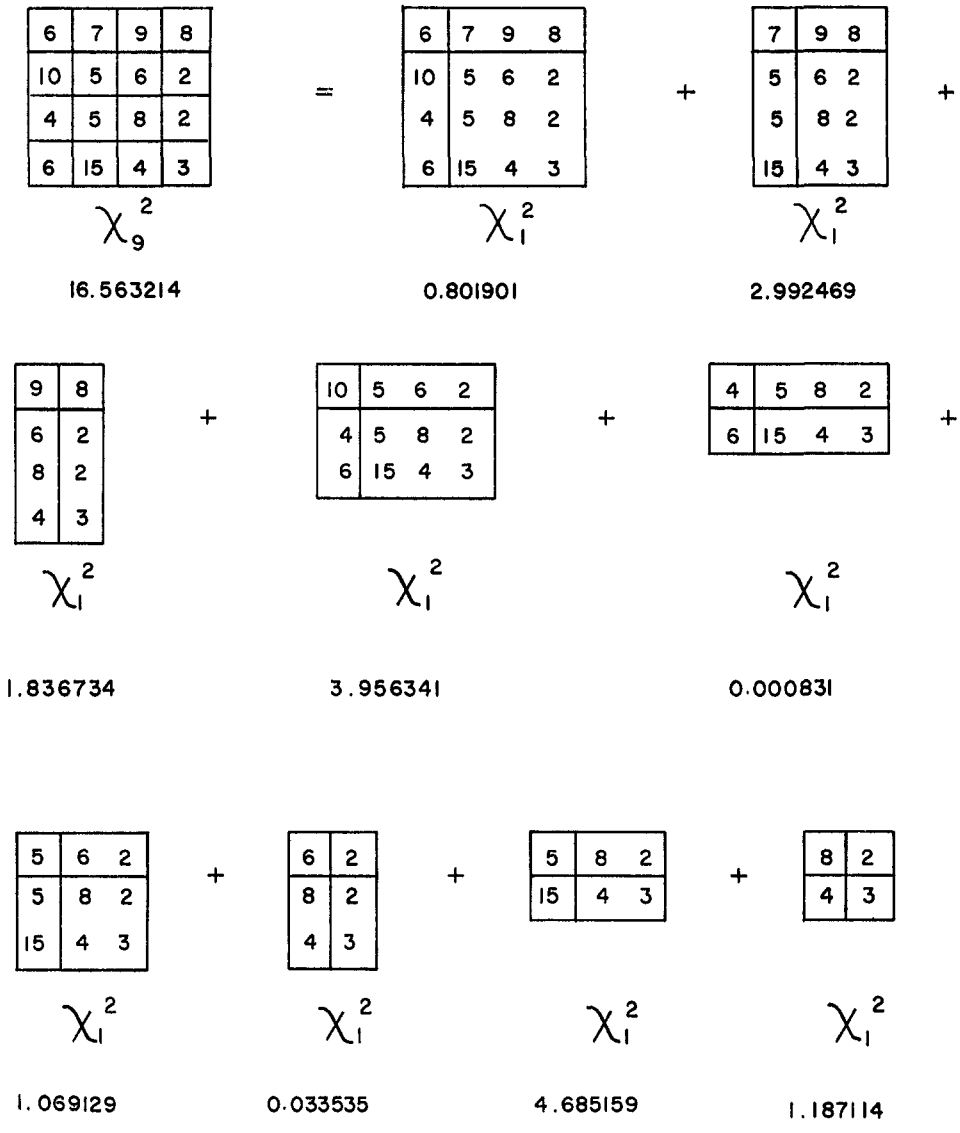

FIG. 3a. Table from Figure 1, with numbers substituted for symbols.

not exist when both dimensions have been reduced. Nevertheless, it should be apparent that reducing only the rows or only the columns may not provide sufficient information. It shall now be shown that these reductions are only first steps in a complete analysis.

Additional Applications and Advantages of the Alternative Equation

The expression

$$\sum \frac{(f_o - f_e)^2}{F_e}$$

is easier to calculate than Irwin's equivalent formula because in a 2×2 table all the squared deviations are equal. Apart from convenience, however, this formula has the important advantage of not being restricted to 2×2 tables. This fact enables it to have an application not directly available by Irwin's formula. If a researcher's interest lies in a particular configuration consisting of all the rows (or columns) of the original table, with one or more of the columns (or rows) eliminated, then for this partition considered as a

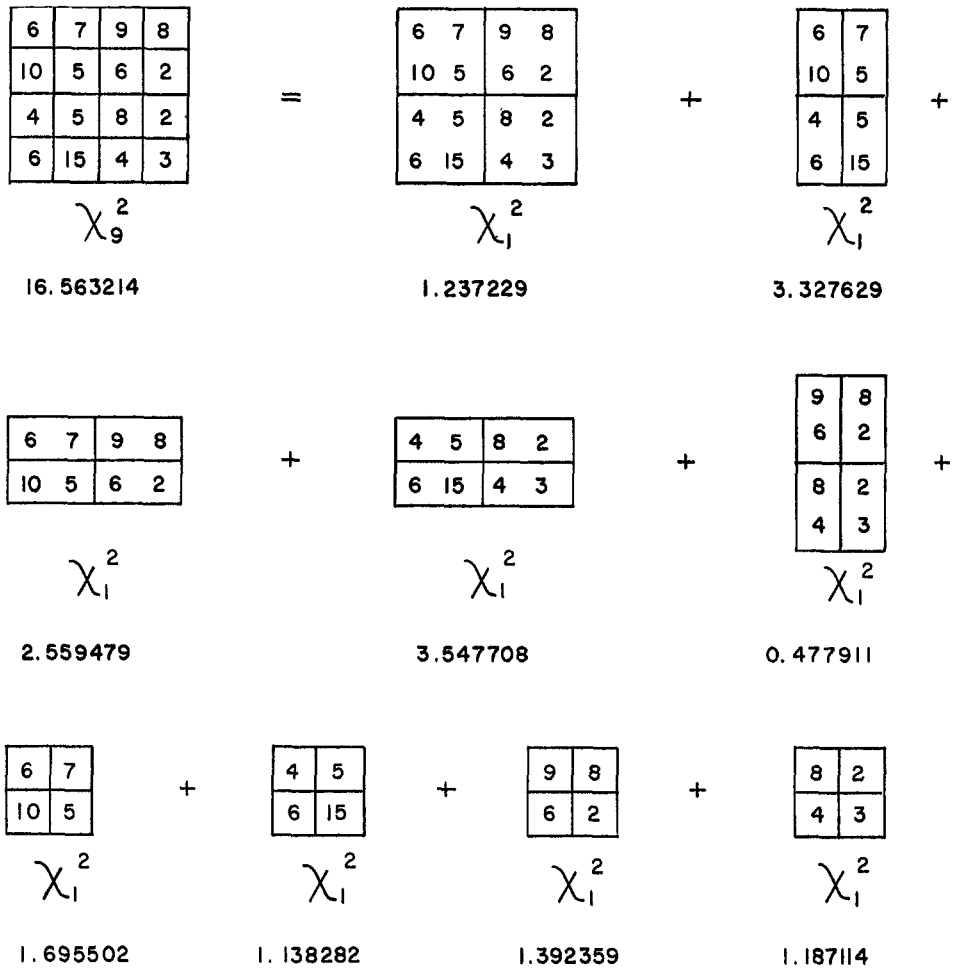

FIG. 3b. Table from Figure 2, with numbers substituted for symbols.

FIG. 3. Examples of partitioning with chi-square values calculated.

whole, that is, not merely as a 2 × 2 partition, chi-square can be calculated by this formula. Using Irwin's formula, the chi-square would have to be found by calculating all the 2 × 2 partitions which contain in single form elements not included in the partition of interest, and subtracting the sum of the values from the chi-square of the whole table.

As an illustration of this principle the first row has been removed from the 4 × 4 table of Figure 1, and the χ_1^2 calculated for Partitions 1, 2, and 3, and χ_6^2 calculated for Partition 4. Similarly, the first two rows have been removed and the χ_1^2 calculated for Partitions 1, 2, 3, 4, 6, and 7, and χ_3^2 calculated for Partition 5. The results are shown in Figure 4a and 4b, respectively.

Equation 5 has no apparent advantage over Kastenbaum's Equation 2, other than ease of interpretation and conceptualization; however, if the notation is changed and the terms rearranged, a general equation for any partition of any mode or degree can be derived.

Let l = number of rows in partitioned table; m = number of columns in partitioned table; r = number of rows in whole table; c = number of columns in whole table; n_{ij} = observed frequency in cell ij ($i = 1$ to l to r, $j = 1$ to m to c);

$$n_{i.} = \sum_{j=1}^{m} n_{ij}$$

$$n_{.j} = \sum_{i=1}^{l} n_{ij}$$

10	5	6	2
4	5	8	2
6	15	4	3

$\chi_1^2 = 0.801901$

$\chi_1^2 = 2.992469$

$\chi_1^2 = 1.836734$

$\chi_6^2 = 10.932109$

$\chi_9^2 = 16.563213$

FIG. 4a. Table for six degrees of freedom remaining after the first row has been removed from the 4×4 table of Figure 3, and the chi-square values for each of the partitions.

4	5	8	2
6	15	4	3

$\chi_1^2 = 0.801901$

$\chi_1^2 = 2.992469$

$\chi_1^2 = 1.836734$

$\chi_1^2 = 3.956341$

$\chi_1^2 = 1.069129$

$\chi_1^2 = 0.033535$

$\chi_3^2 = 5.873104$

$\chi_9^2 = 16.563213$

FIG. 4b. Table for three degrees of freedom remaining after the first two rows have been removed from the 4×4 table from Figure 3, and the chi-square values for each of the partitions.

FIG. 4. Examples of partitioning in which some of the components have more than one degree of freedom.

e_{ij} = expected frequency for cell ij calculated from the margins of the original table

$$e_{i.} = \sum_{j=1}^{m} e_{ij}$$

$$e_{.j} = \sum_{i=1}^{l} e_{ij}$$

Then Equation 5, when only the number of rows has been reduced, may be written as

$$\chi_{(l-1)(c-1)}^2 = \sum_{i=1}^{l} \sum_{j=1}^{c} \frac{\left[n_{ij} - \dfrac{\sum_{i=1}^{l} n_{ij}}{\sum_{i=1}^{l} n_{i.}} \right]^2}{e_{ij}}$$

and by algebra

$$\chi_{(l-1)(c-1)}^2 = \sum_{i=1}^{l} \sum_{j=1}^{c} \frac{n_{ij}^2}{e_{ij}} - \sum_{j=1}^{c} \frac{(\sum_{i=1}^{l} n_{ij})^2}{e_{.j}} \qquad [6]$$

for the partitioned table. Similarly, if only the number of columns has been reduced, Equation 5 may be written as

$$\chi_{(r-1)(m-1)}^2 = \sum_{i=1}^{r} \sum_{j=1}^{m} \frac{n_{ij}^2}{e_{ij}} - \sum_{i=1}^{r} \frac{(\sum_{j=1}^{m} n_{ij})^2}{e_{i.}} \qquad [7]$$

for the partitioned table. Consider now that chi-square for the whole table can be written in the commonly used form

$$\chi_{(r-1)(c-1)}^2 = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{n_{ij}^2}{e_{ij}} - N \qquad [8]$$

Therefore, the only differences between Equations 6, 7, and 8 are the last terms. Equation 8 has N, Equation 6 has the square of the observed value of each attenuated column divided by its original expectation, and Equation 7 has the square of the observed value of each attenuated row divided by its original expectation. That is, Equations 6 and 7 take into account the deviations from proportionality, respectively, by considering the expected and observed values of the remaining portions of the columns and rows.

Now, consider the possibility of obtaining an equation for chi-square when both the number of rows and the number of columns have been reduced, that is, both columns and rows have been attenuated. It can be shown by the most elementary algebra, although laboriously lengthy, that

$$\chi_{(l-1)(m-1)}^2 = \sum_{i=1}^{l} \sum_{j=1}^{m} \frac{n_{ij}^2}{e_{ij}} - \sum_{i=1}^{l} \frac{(\sum_{j=1}^{m} n_{ij})^2}{e_{i.}} - \sum_{j=1}^{m} \frac{(\sum_{i=1}^{l} n_{ij})^2}{e_{.j}} + \frac{(\sum_{i=1}^{l} \sum_{j=1}^{m} n_{ij})^2}{\sum_{i=1}^{l} \sum_{j=1}^{m} e_{ij}} \qquad [9]$$

Let $o_{i.}$ and $o_{.j}$ be the observed margin totals in the partitioned table, and O be the total observed frequency in the partitioned table for which E is the total expected frequency; then Equation 9 becomes

$$\chi_{(l-1)(m-1)}^2 = \sum_{i=1}^{l}\sum_{j=1}^{m}\frac{n_{ij}^2}{e_{ij}} - \sum_{i=1}^{l}\frac{o_{i.}^2}{e_{i.}} - \sum_{j=1}^{m}\frac{o_{.j}^2}{e_{.j}} + \frac{O^2}{E} \quad [10]$$

where,

$$o_{i.} = \sum_{j=1}^{m} n_{ij}$$

$$o_{.j} = \sum_{i=1}^{l} n_{ij}$$

$$O = \sum_{i=1}^{l}\sum_{j=1}^{m} n_{ij}$$

$$E = \sum_{i=1}^{l}\sum_{j=1}^{m} e_{ij}$$

All previous equations for chi-square are special cases of this general equation. Even the standard Equation 1 is the special case of exhaustive categories. It is easily shown that Equation 10 reduces to the other equations when the appropriate special conditions are met. This equation may also be written in the following form, which has interesting properties, conceptually,

$$\chi_{(l-1)(m-1)}^2 = \sum_{i=1}^{l}\sum_{j=1}^{m}\frac{(n_{ij}-e_{ij})^2}{e_{ij}} - \sum_{i=1}^{l}\frac{(o_{i.}-e_{i.})^2}{e_{i.}} - \sum_{j=1}^{m}\frac{(o_{.j}-e_{.j})^2}{e_{.j}} + \frac{(O-E)^2}{E} \quad [11]$$

Equation 10 may, therefore, be considered as a test of independence involving (*a*) the squared deviations of observation from expectation divided by expectation, for each cell; (*b*) a correction or subtraction out of the squared deviations of observation from expectation divided by expectation, for each margin total; and (*c*) a further correction for having counted the total observed deviation from expectation twice. That is, it is a test of independence corrected for lack of goodness-of-fit in the margins.

Implications of the General Equation

An additional application of the equation derived in this paper,

$$\chi_{(l-1)(m-1)}^2 = \sum_{i=1}^{l}\sum_{j=1}^{m}\frac{n_{ij}^2}{e_{ij}} - \sum_{i=1}^{l}\frac{o_{i.}^2}{e_{i.}} - \sum_{j=1}^{m}\frac{o_{.j}^2}{e_{.j}} + \frac{O^2}{E} \quad [10]$$

arises in situations where two assumptions of the chi-square test of independence have not been satisfied. These common violations involve the pooling of data to obtain desired minimum expectations, and the failure to meet the requirement of exhaustive categories. Lancaster (1950) had previously noted the relationship between partitioning and pooling; the relationships among partitioning, pooling, and nonexhaustiveness can now be explicitly determined with the above equation.

Various investigators have consistently cautioned against the use of small cell entries, some setting the desired minimum expectation at 5 (e.g., Cochran, 1954), or more conservatively at 10 (e.g., Lewis & Burke, 1949). In order to follow this advice, a researcher must often resort to the alternatives of either discarding the categories which do not have minimum expectations, or pooling these categories with adjacent categories. Both of these alternatives are inadequate since they provide a way of achieving desired minimum expectations at the expense of introducing other, and perhaps equally serious, errors.

When data are pooled, error is usually introduced, since only in trivial cases are the relationships of the marginal totals not altered. Moreover, for chi-square, the categories often do not have any natural ordering, so that choosing the categories with which the small cell frequencies will be combined becomes an arbitrary decision, accompanied by a spurious lowering or raising of the chi-square value. There are certain situations where the cate-

TABLE 1

	Ps	OT	CC
Af	30	102	28
Al	48	23	20
Or	19	80	75
Sc	121	344	382
Se	18	11	141

gories do follow a natural ordering, and pooling may be a legitimate operation, as in goodness-of-fit tests. There are also numerous occasions when an ordering seems apparent but is, in fact, imposed by the researcher and not the natural events.

The alternative procedure of discarding categories is patently an inefficient procedure since all data should be utilized. More important, the discarding of categories fails to satisfy the requirement of exhaustiveness.

The errors caused by either the pooling of data or the use of nonexhaustive categories can be eliminated by the technique suggested in this paper. That is, if one or more rows and/or columns consist of cells which do not have desired minimum expectations, instead of discarding these rows and/or columns, or pooling them with adjacent entries, a chi-square value should be calculated for the table remaining by Equation 10. This value can be regarded as the contribution of the table remaining to the chi-square of the original table. The degrees of freedom for this table are the

TABLE 2

	Ps	OT	CC	
Af	34.4	167.4	10.9	160.0
Al	154.7	15.0	9.8	91.0
Or	12.7	94.7	72.2	174.0
Sc	105.6	359.8	384.6	847.0
Se	11.6	1.8	261.0	170.0
	236.0	560.0	646.0	1142.0

$\chi_4^2 = 254\,254660$, $p < .001$

TABLE 3

	Ps	OT	CC	
Or + Sc	117.3	453.4	456.6	1021.0
Af + Al + Se	133.8	113.1	189.4	421.0
	236.0	560.0	646.0	1442.0

$\chi_2^2 = 21.583895$, $p < .001$

same as they would be if the small expectations were pooled or discarded completely, but the value of the statistic is now exact.[2]

It is interesting to note that although the requirement of exhaustiveness has been repeatedly mentioned in the literature, it has not been elaborated upon and appears to have been treated more as a theoretical ideal than one to be sought in practice. Perhaps this cursory treatment of the problem in the past indicated the lack of a satisfactory solution.

An Example

It is perhaps useful to present a real-life example to demonstrate the utility of these techniques and the manner in which the general equation for chi-square facilitates the computation. The example to be used is Table 35 from Hollingshead and Redlich (1958, p. 288). It shall be assumed that the sample of 1,442 constituted a random sample from the population of psychotic patients. Each subject was categorized according to the diagnostic groups

[2] It should be noted that in some sense no computed value of chi-square is exact since the computational formulas yield values which are only approximately distributed as chi-square. That is, chi-square is approached as the number of frequency entries approaches infinity. In this sense, the components add up exactly to an approximation.

TABLE 4

	Ps	OT	CC	
Or	12.7	94.7	72.2	174.0
Sc	105.6	359.8	384.6	847.0
	117.3	453.4	456.6	1021.0

$\chi_2^2 = 2.196957$, $p \cong .33$

TABLE 5

	Ps + OT	CC	
Af	90.0	10.9	160.0
Al	100.3	9.8	91.0
Se	9.0	261.0	170.0
	231.6	189.4	421.0

$\chi_2^2 = 60.116576, p < .001$

affective psychoses (Af), alcoholic psychoses (Al), organic psychoses (Or), schizophrenic psychoses (Sc), senile psychoses (Se); and treatments psychotherapy (Ps), organic therapy (OT), custodial care (CC). The analysis is done using Equation 10. The term $\frac{(\text{observed})^2}{(\text{expected})}$ is given for each cell and each margin in the tables below, even when the terms are redundant or corresponding terms sum to zero. To further facilitate the readers' following the computations, the individual terms are shown to only one decimal place, although the chi-square values are computed to six decimal places.

Table 1 shows Hollingshead and Redlich's percentage data converted to frequencies. Table 2 shows that the original table yields a chi-square for eight degrees of freedom equal to 254.254660. (The chi-square value in the original text obviously contains an inconsequential computational error.) Table 3 shows the chi-square calculation for the diagnostic groups Or and Sc versus Al, Af, and Se. It can be seen that these two sets of diagnostic groups and the treatment categories are not independent. Table 4 shows, however, that

TABLE 6

	Ps	OT	
Af	34.4	167.4	90.0
Al + Se	102.0	11.4	69.4
	133.8	113.1	231.6

$\chi_1^2 = 140.513219, p < .001$

TABLE 7

	Ps	OT	
Al	154.7	15.0	100.3
Se	11.6	1.8	9.0
	102.0	11.4	69.4

$\chi_1^2 = 29.844013, p < .001$

diagnostic categories Or and Sc and the treatment categories must be considered independent. Table 5 shows that the diagnostic categories Af, Al, and Se are not independent of the treatment categories CC versus Ps plus OT combined. Table 6 shows that Af versus Al plus Se combined are not independent of Ps and OT. Table 7 shows that Al and Se are not independent of Ps and OT.

Although the overall chi-square for eight degrees of freedom is a very large number, it may be concluded that the diagnostic categories of organic psychoses and schizophrenic psychoses do not significantly differ from each other in terms of the proportion of patients given psychotherapy, organic therapy, or custodial care. The organic and schizophrenic psychoses groups differ from the three other diagnostic categories in the proportion of patients given the three types of treatment, but the organic and schizophrenic groups do not differ from each other. All other tests of independence are significant.

Although this example involves neither the problem of small expected frequencies or originally nonexhaustive categories, it is an interesting illustration. The overall analysis yields what looks to be an exceedingly "strong" relationship; but on a more detailed analysis, the relationship fails to exist where one might logically expect to find it.

REFERENCES

CASTELLAN, N. J., JR. On the partitioning of contingency tables. *Psychological Bulletin*, 1965, **64**, 330–338.

COCHRAN, W. G. Some methods for strengthening the common χ^2 tests. *Biometrics*, 1954, **10**, 417–451.

FISHER, R. W. *Statistical methods for research workers*, (4th ed.). Edinburgh: Oliver & Boyd, 1932.

HOLLINGSHEAD, A. B., & REDLICH, F. C. *Social class and mental illness: A community study*. New York: Wiley, 1958.

IRWIN, J. O. A note on the subdivision of χ^2 into components. *Biometrika*, 1949, **36**, 130–134.

KASTENBAUM, M. A. A note on the additive partitioning of chi-square in contingency tables. *Biometrics*, 1960, **16**, 416–422.

KIMBALL, A. W. Short-cut formulas for the exact partition of χ^2 in contingency tables. *Biometrics*, 1954, **10**, 452–458.

LANCASTER, H. O. The derivation and partition of χ^2 in certain discrete distributions. *Biometrika*, 1949, **36**, 117–129.

LANCASTER, H. O. The exact partition of χ^2 and its application to the problem of the pooling of small expectations. *Biometrika*, 1950, **37**, 267–270.

LEWIS, D., & BURKE, C. J. The use and misuse of the chi-square test. *Psychological Bulletin*, 1949, **46**, 433–489.

MAXWELL, A. E. *Analysing qualitative data*. London: Methuen, 1961.

SNEDECOR, G. W. *Statistical methods*. (5th ed.). Ames, Iowa: Iowa State College Press, 1956.

WALKER, H. M., & LEV, J. *Statistical inference*. New York: Holt, 1953.

(Received July 6, 1965)