MATH 307-001 - December 1, 2003 - Instructor: Candace Kent
SOLUTIONS to HW # 3
On Sections 11.1, 11.2, and 11.3

HW # 3 consists of a total of twenty-five problems. Each problem is worth 5 points: I have stated in the past that you will receive the full 5 points for a problem if you show that you attempted to do the problem (successfully or not); and you will receive 0 points if you show absolutely no work beyond, say, rewriting the statement of the problem. However, given the particular difficulty with this homework assignment, I may be much more flexible than I stated in the past.

Remember, you are allowed to use calculators or computer software to make numerical calculations, assist in graphing equations, or simply check your work, but unless otherwise stated, you should do the work BY HAND.

Section 11.1: 16, 18, 38, 40.

Section 11.2: 6, 8, 12, 26, 32.

Section 11.3: 14, 16, 18, 20, 22, 26, 28, 30,
32, 36, 40, 48, 50, 52, 54, 56.

16. $f(x,y) = x^2 - y^2$

We first need some definitions and remarks:

DEFINITION 1. Let $f(x,y)$ be a function of two variables. Then the LEVEL CURVES of f are the curves with the equations

$$f(x,y) = k, \quad k = \text{constant} \in \text{Range of } f.$$

Note that the LEVEL CURVES are the HORIZONTAL TRACES of the graph of f (a surface with equation $z = f(x,y)$) in the horizontal plane $z = k$ projected down/up to the xy-plane.

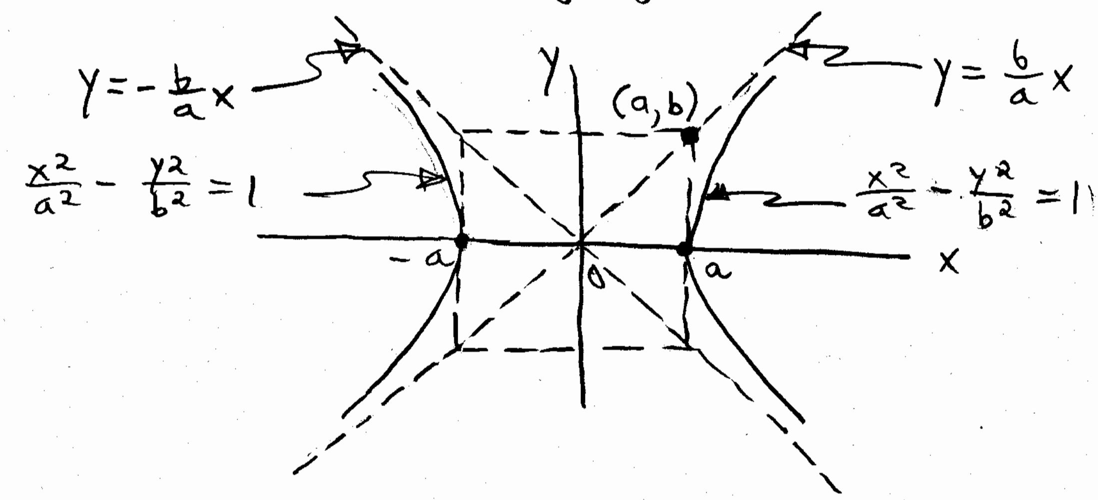
DEFINITION 2. Let $f(x,y)$ be a function of two variables. Then a CONTOUR MAP of f is the set of LEVEL CURVES in the xy-plane for a certain set of values of k.

REMARK 1. The normal form of a hyperbola (where the coordinate axes coincide with the axes of the hyperbola) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(Problem 16 continued)

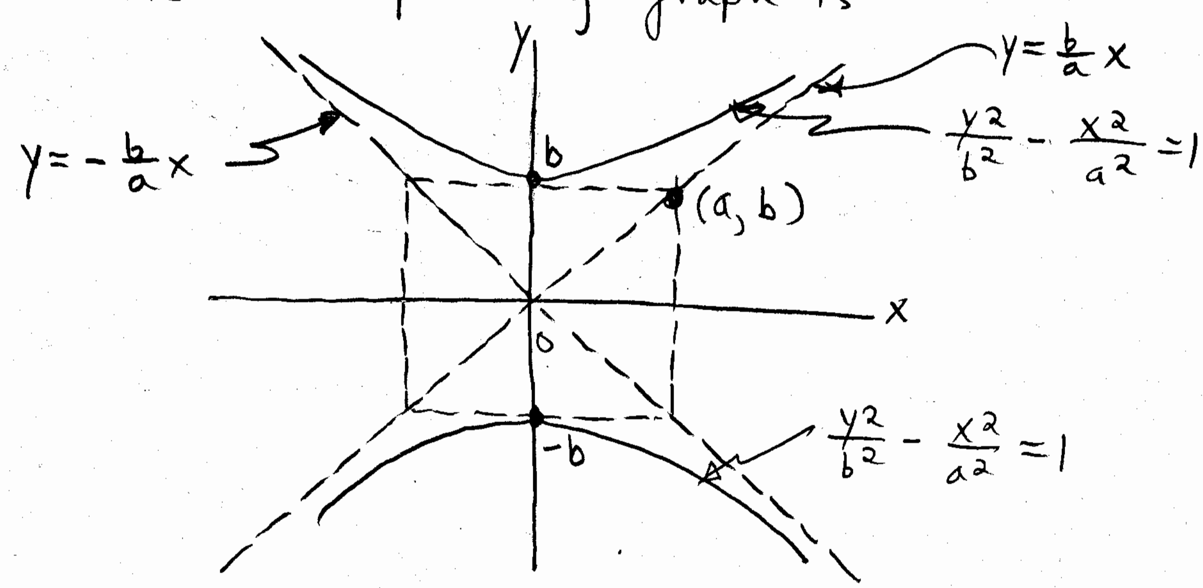
and its corresponding graph is



REMARK 2. The conjugate of the normal form of a hyperbola is

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

and its corresponding graph is



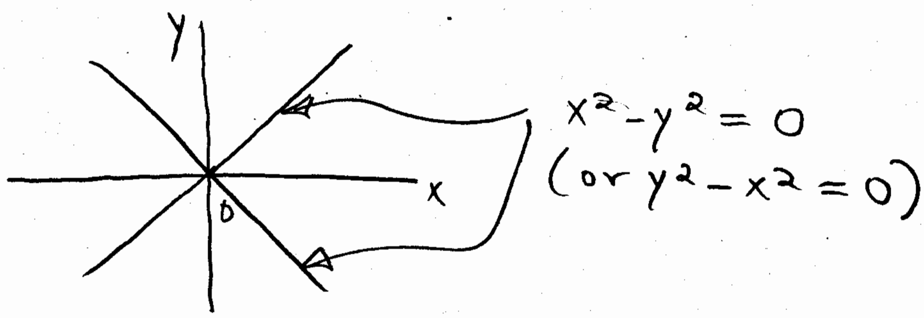
(Problem 16 continued)

REMARK 3. The graph of

$$x^2-y^2=0 \quad (\text{or } y^2-x^2=0)$$

is the union of the graphs of the equations

$$y=x, \quad y=-x.$$



REMARK 4. A rectangular hyperbola has equal axes, $a=b$, and its equation is

$$x^2-y^2=a^2.$$

The asymptotes are perpendicular.

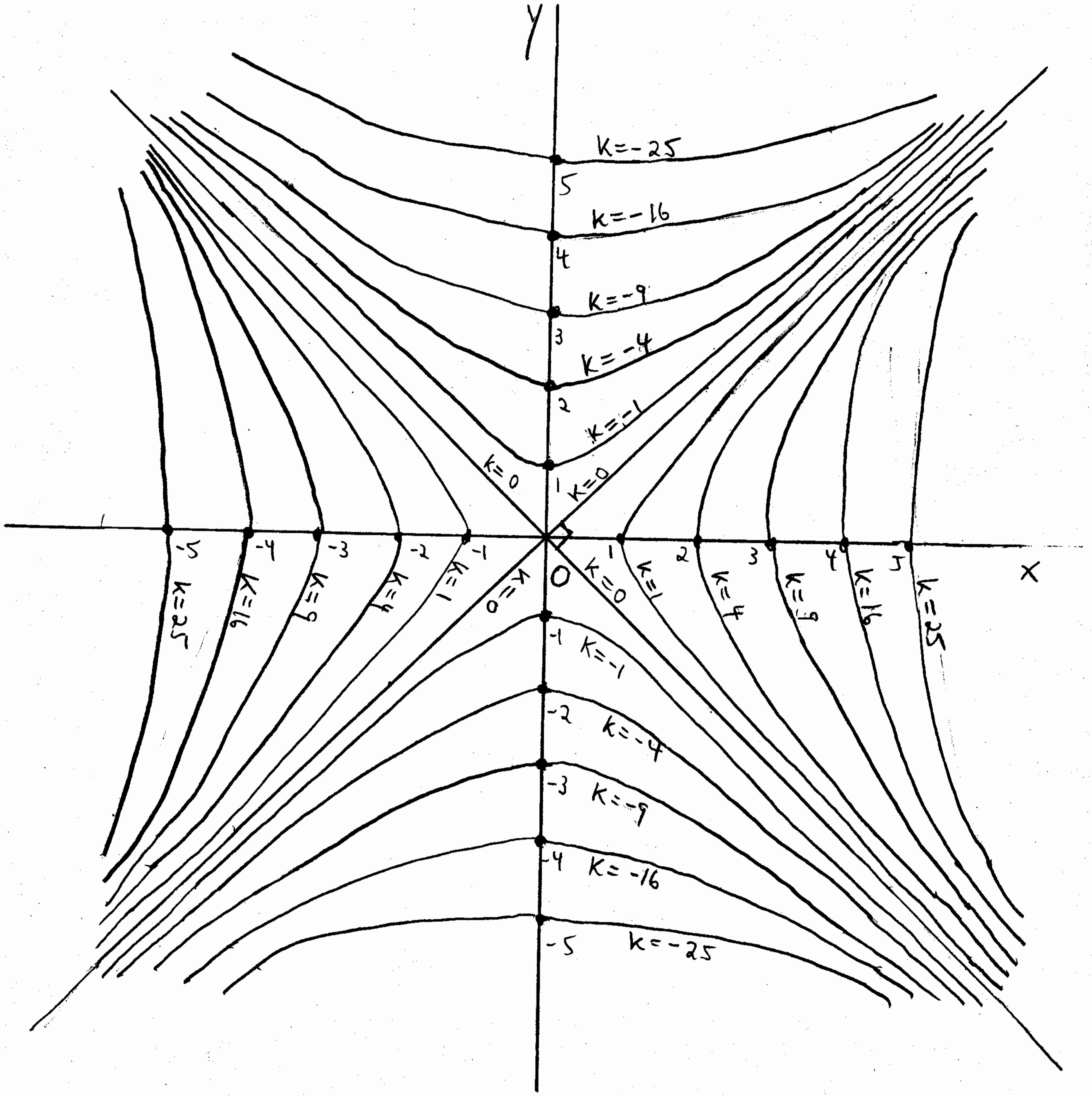
o

We sketch the LEVEL CURVES of $f(x,y)=x^2-y^2$ whose equations are

$$x^2-y^2=k$$

where $k=0, \pm 1, \pm 4, \pm 9, \pm 25$:

(Problem 16 continued)



CONTOUR MAP OF $f(x,y)=x^2-y^2$

18. $f(x,y) = e^{y/x}$.

We first need a remark.

REMARK. If

$$e^{y/x} = k, \qquad k = \text{constant} > 0,$$

then we have

$$\ln e^{y/x} = \ln k \Rightarrow \frac{y}{x} = \ln k \Rightarrow$$

$$y = mx, \qquad m = \ln k \in \mathbb{R}$$

where

1. $m < 0$ if $0 < k < 1$,
2. $m = 0$ if $k = 1$,
3. $m > 0$ if $k > 1$.

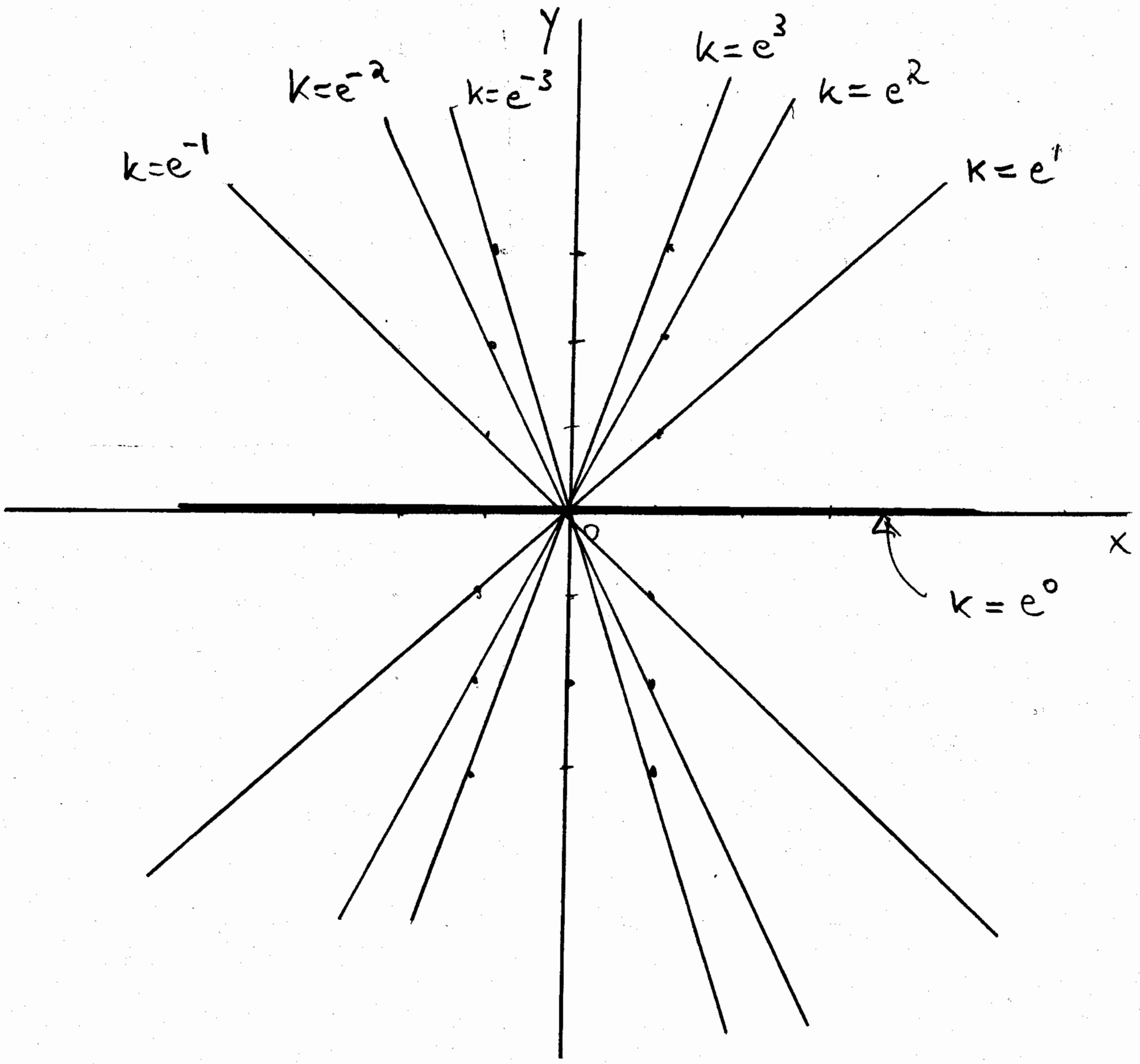
We sketch the LEVEL CURVES of $f(x,y) = e^{y/x}$ whose equations are

$$e^{y/x} = k \quad (\text{or } y = mx, \ m = \ln k)$$

where $k = e^{-3}, e^{-2}, e^{-1}, e^{0}, e^{1}, e^{2}, e^{3}$

(Problem 18 continued)

(or $m = -3, -2, -1, 0, 1, 2, 3$, respectively):



CONTOUR MAP OF $f(x,y) = e^{y/x}$

38. $f(x, y, z) = x^2 + 3y^2 + 5z^2$

The LEVEL SURFACES ("SURFACES" and not "CURVES" since here we are dealing with a function of 3, not 2, variables) are given by the equations

$$f(x, y, z) = k, \quad k \geq 0 \implies$$

since $x^2+3y^2+5z^2 \geq 0$ for all (x, y, z)

$$x^2 + 3y^2 + 5z^2 = k, \quad k \geq 0 \implies$$

① $k > 0$.

$$\frac{1}{k}(x^2 + 3y^2 + 5z^2) = \frac{1}{k}(k) \implies$$

$$\frac{x^2}{(\sqrt{k})^2} + \frac{y^2}{\left(\sqrt{\frac{k}{3}}\right)^2} + \frac{z^2}{\left(\sqrt{\frac{k}{5}}\right)^2} = 1.$$

According to TABLE on p. 691 of your text, this equation describes a family of $\boxed{\text{ELLIPSOIDS}}$

② $k = 0$.

$$x^2 + 3y^2 + 5z^2 = 0 \implies$$

$$x = y = z = 0.$$

Thus, one (DEGENERATE) LEVEL SURFACE is the $\boxed{\text{ORIGIN}}$.

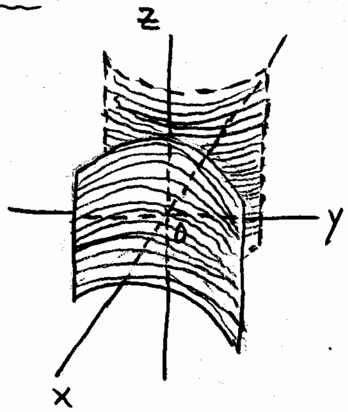
40. $f(x,y,z) = x^2 - y^2$

The LEVEL SURFACES ("SURFACES" and not "CURVES" since here we are dealing with a function of 3, not 2, variables) are given by the equations

$$f(x,y,z) = k, \quad k \text{ any real} \Rightarrow$$

$$x^2 - y^2 = k, \quad k \text{ any real} \Rightarrow$$

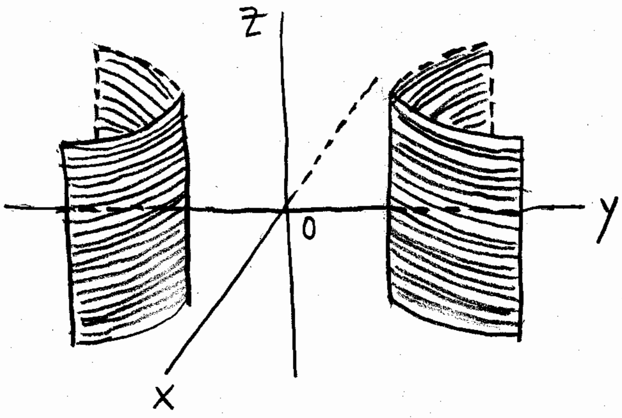
① k > 0.



So LEVEL SURFACES form a family of

HYPERBOLIC CYLINDERS along z-axis

② k < 0.



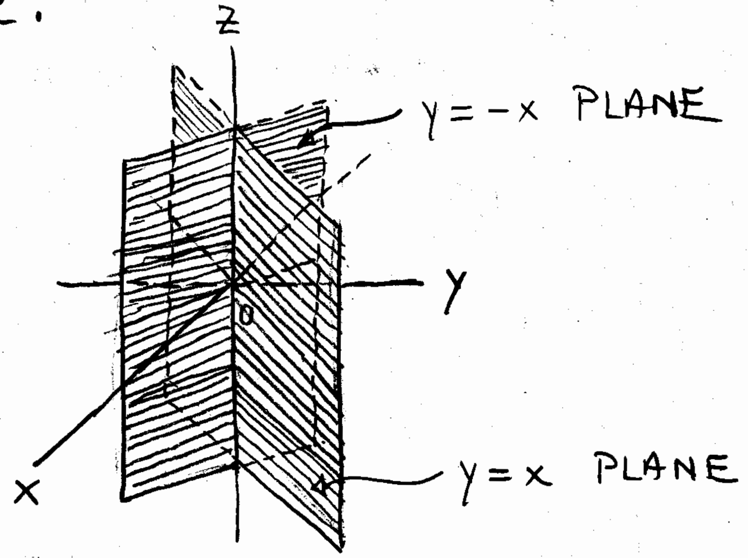
So LEVEL SURFACES form a family of

HYPERBOLIC CYLINDERS along y-axis where

$x^2 - y^2 = -k > 0$

(Problem 40 continued)

(3) $k=0$.



So LEVEL SURFACE is the

$\boxed{\text{INTERSECTION of the } y=x \text{ and } y=-x \text{ PLANE.}}$

6. $\lim\limits_{(x,y)\to(6,3)} xy\cos(x-2y)$

With any limit, we first try substituting $(x,y)=(6,3)$ directly into the function

$$f(x,y) = xy\cos(x-2y).$$

If we do not obtain an undefined or indeterminate result like

$$\frac{\text{a number}}{0},\ \frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 0^{\infty},\ \text{etc.},$$

then we call the result our limit. This is the case here:

$$\lim_{(x,y)\to(6,3)} xy\cos(x-2y) = 6\cdot 3\cdot\cos(6-2\cdot 3)$$

plug in

$$= 18\cdot\cos 0$$

$$= 18\cdot 1$$

$$= \boxed{18}$$

8. $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

We first try plugging $(x,y)=(0,0)$ directly into $f(x,y)=\frac{(x+y)^2}{x^2+y^2}$:

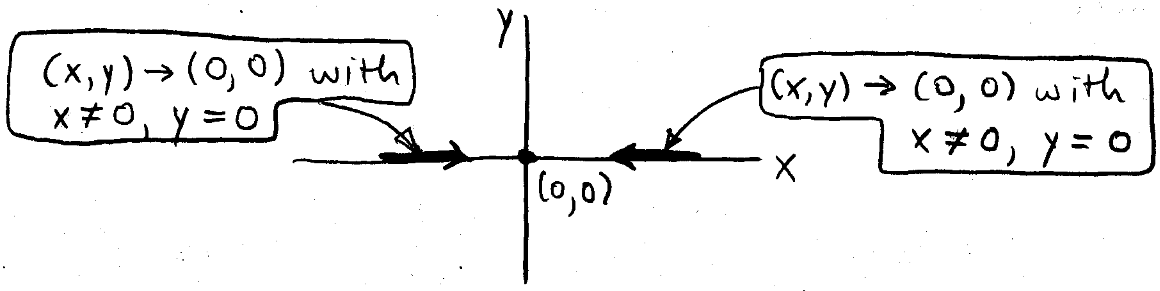
$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} = \frac{(0+0)^2}{0^2+0^2} = \frac{0}{0}.$$

Since we obtain an indeterminate form, $\frac{0}{0}$, we must try to use another method (like application of the SQUEEZE THEOREM) or show that our limit DOES NOT EXIST (DNE).

We will first try to show our limit DNE by considering the limit of $f(x,y)=\frac{(x+y)^2}{x^2+y^2}$ over two different paths taken in the xy-plane and obtaining two different values of the limit (in order for a limit to be said to exist, it must be equal to the same value over all paths

(Problem 8 continued)

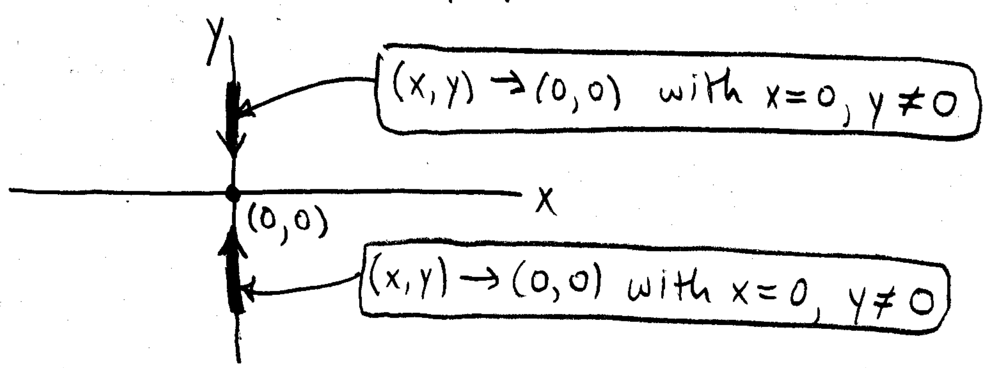
$\boxed{\text{PATH \#1}}$: (x,y) approaches $(0,0)$ along the x-axis in the xy-plane.



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#1}}} \frac{(x+y)^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#1}}} \frac{(x+0)^2}{x^2+0^2}$$

$$= \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#1}}} \frac{x^2}{x^2} = \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#1}}} 1 = \boxed{1 \equiv L_1}$$

$\boxed{\text{PATH \#2}}$: (x,y) approaches $(0,0)$ along the y-axis in the xy-plane.



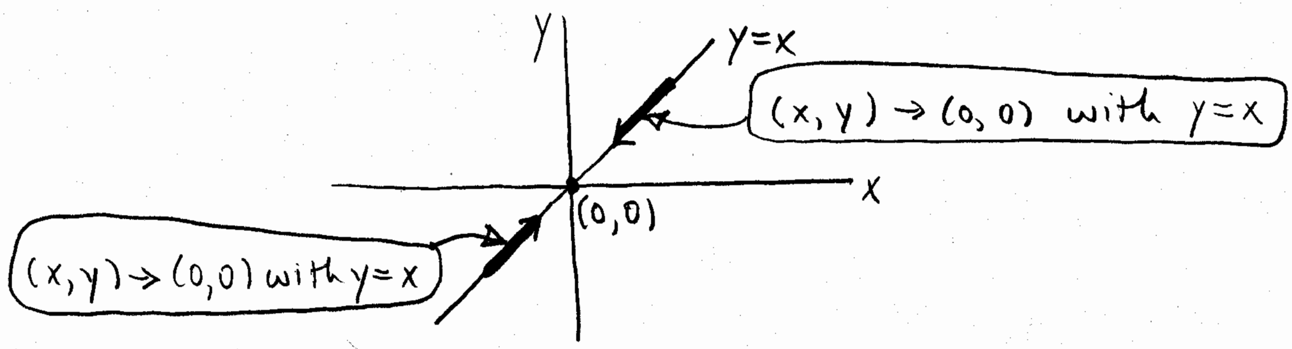
(Problem 8 continued)

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#2}}} \frac{(x+y)^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#2}}} \frac{(0+y)^2}{0^2+y^2}$$

$$= \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#2}}} \frac{y^2}{y^2} = \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#2}}} 1 = \boxed{1 \equiv L_2}$$

Since $L_1 = 1 = L_2$, we have not yet shown that our limit DNE, and for all we know, our limit may very well exist, although we have not proven that either. We try a 3^{rd} "path":

$\boxed{\text{PATH \#3}}$: (x, y) approaches $(0, 0)$ along the diagonal line $y = x$ in the xy-plane.



(Problem 8 continued)

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#3}}} \frac{(x+y)^2}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#3}}} \frac{(y+y)^2}{y^2+y^2}$$

$$= \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#3}}} \frac{4\cancel{y^2}}{2\cancel{y^2}} = \lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#3}}} 2 = \boxed{2 \equiv L_3}$$

Since

$$L_1 = 1 \neq 2 = L_3,$$

our limit

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} \quad \text{DNE}}$$

12. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 \sin^2 y}{x^2+2y^2}$

We first try plugging $(x,y)=(0,0)$ directly into $f(x,y) = \dfrac{x^2 \sin^2 y}{x^2+2y^2}$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+y^2} = \frac{0^2 \sin^2 0}{0^2+0^2} = \frac{0}{0}$$

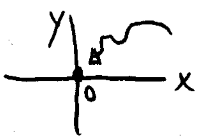
Since we obtain an indeterminate form, $\frac{0}{0}$, we must try to use another method (like application of the SQUEEZE THEOREM) or show that our limit DOES NOT EXIST (DNE).

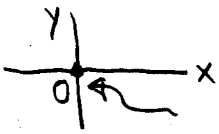
We will first try to show our limit DNE by considering the limit of $f(x,y) = \dfrac{x^2 \sin^2 y}{x^2+2y^2}$ over at most three known paths that are different and see if we obtain three different values of the limit (in order for a limit to be said to exist, it must be equal to the same value over all paths taken in the xy-plane). If we obtain the same value of the limit

(Problem 8 continued)

taken in the xy-plane):

$$\left.\begin{array}{l} \displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#1}}} \frac{(x+y)^2}{x^2+y^2} = L_1 \\[2em] \displaystyle\lim_{\substack{(x,y)\to(0,0)\\ \text{PATH \#2}}} \frac{(x+y)^2}{x^2+y^2} = L_2 \end{array}\right\}$$





We want $L_1 \neq L_2$ for limit not to exist.

So, we consider $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$ over the the following two paths in the xy-plane that are easy for us to describe, and hope that we obtain two different values of the limit:

(**Problem 12 continued**)

over the three paths, we will turn to trying to show the limit does in fact exist using the SQUEEZE THEOREM. If that does not work for us, we will have no idea if our limit exists or not.

① $$\lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along x-axis} \\ (x\neq 0,\, y=0)}} \frac{x^2\sin^2 y}{x^2+2y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along x-axis} \\ (x\neq 0,\, y=0)}} \frac{x^2\sin^2 0}{x^2+2\cdot 0^2}$$

$$= \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along x-axis} \\ (x\neq 0,\, y=0)}} \frac{0}{x^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along x-axis} \\ (x\neq 0,\, y=0)}} 0 = \boxed{0}$$

② $$\lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along y-axis} \\ (x=0,\, y\neq 0)}} \frac{x^2\sin^2 y}{x^2+2y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along y-axis} \\ (x=0,\, y\neq 0)}} \frac{0^2\sin^2 y}{0^2+2y^2}$$

$$= \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along y-axis} \\ (x=0,\, y\neq 0)}} \frac{0}{2y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along y-axis} \\ (x=0,\, y\neq 0)}} 0 = \boxed{0}$$

(Problem 12 continued)

(3) $$\lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along diagonal} \\ (y=x;\ x\neq 0,\ y\neq 0)}} \frac{x^2\sin^2 y}{x^2+2y^2} = \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along diagonal} \\ (y=x,\ x\neq 0,\ y\neq 0)}} \frac{y^2\sin^2 y}{y^2+2y^2}$$

$$= \lim_{\substack{(x,y)\to(0,0) \\ \text{PATH along diagonal} \\ (y=x,\ x\neq 0,\ y\neq 0)}} \frac{\sin^2 y}{3} = \frac{\sin^2 0}{3} = \frac{0}{3} = \boxed{0}$$

Now can plug $(x,y)=(0,0)$ directly into $\frac{\sin^2 y}{3}$

Since we did not get at least two different values of the limit along two different (but known and easy-to-work-with) paths, we cannot show the limit DNE.

We now will try our luck at showing the limit in this problem exists using the SQUEEZE THEOREM.

(**Problem 12 continued**)

First of all, observe that for

① $x \neq 0, y \neq 0$, or

② $x = 0, y \neq 0$, or

③ $x \neq 0, y = 0$

(but never both $x=0, y=0$), we have the following:

$$0 \leq \frac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y$$

where the first inequality should be obvious from the fact that $x \geq 0, y \geq 0$ (but x and y are never both 0) and the second inequality comes about in this way:

$$\frac{x^2 \sin^2 y}{x^2+2y^2} \leq \frac{\cancel{x^2} \sin^2 y}{\cancel{x^2}} = \sin^2 y.$$

since $x^2 + 2y^2 \geq x^2$ for $x \geq 0, y \geq 0$

(Problem 12 continued)

Next observe that

$$\lim_{(x,y)\to(0,0)} 0 = 0,$$

$$\lim_{(x,y)\to(0,0)} \sin^2 y = \sin^2 0 = 0.$$

Therefore,

$$\underbrace{\lim_{(x,y)\to(0,0)} 0}_{=0} \le \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} \le \underbrace{\lim_{(x,y)\to(0,0)} \sin^2 y}_{=0}$$

which implies that

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} \le 0$$

and so by the SQUEEZE THEOREM our limit exists and

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2} = \boxed{0}.$$

26. $F(x,y) = \dfrac{x-y}{1+x^2+y^2}$

"Short-Cut Rule of Thumb": A RATIONAL FUNCTION (i.e., a function which is a ratio of polynomial functions in x and y) is CONTINUOUS on its DOMAIN, and its DOMAIN is the set of all points (x, y) such that the denominator of the RATIONAL FUNCTION does not equal 0.

So,

$$\text{DOMAIN } F = \{(x,y) \in \mathbb{R}^2 \mid 1+x^2+y^2 \neq 0\}$$
$$= \mathbb{R}^2 \quad \text{since } 1+x^2+y^2 \neq 0 \text{ for all } (x,y) \in \mathbb{R}^2$$

Therefore,

$\boxed{F \text{ is CONTINUOUS on (all of) } \mathbb{R}^2}$

32. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Since $f(x,y) = \dfrac{xy}{x^2+xy+y^2}$ is a RATIONAL FUNCTION on the set $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$ and the denominator of $f(x,y)$ is such that

$$x^2+xy+y^2 \neq 0 \quad \text{on } \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\},$$

then at least

$$\boxed{\begin{array}{l} f(x,y) \text{ is (at least) continuous on the set} \\ \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}. \end{array}}$$

We need now to check to see if $f(x,y)$ is continuous at $(x,y) = (0,0)$ using the "three criteria of continuity" implied by the definition of continuity

(Problem 32 continued)

at a point "$f(x,y)$ is continuous at the point $(x,y)=(a,b)$ if $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$":

① $f(a,b)$ is DEFINED (i.e., $f(a,b) \neq \frac{0}{0}, \frac{a\#}{0}$)

② $\lim_{(x,y)\to(a,b)} f(x,y) = L$ EXISTS

(i.e., L is a single finite value)

③ $L = f(a,b)$ (i.e., $\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b)$).

IF ALL three criteria hold, then $f(x,y)$ is said to be continuous at the point $(x,y)=(a,b)$.

So, we check to see if our $f(x,y)$ satisfies ALL three criteria at the point $(x,y)=(0,0)$:

① $f(0,0) = 0$ (by definition of f) $\Rightarrow$ $f(0,0)$ is DEFINED ✓

(Problem 32 continued)

② $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$

since $(x,y)\to(0,0) \Rightarrow$
$(x,y) \neq (0,0)$

If we plug $(x,y)=(0,0)$ into $\frac{xy}{x^2+xy+y^2}$, we obtain $\frac{0}{0}$, which tells us to turn to another method to determine whether or not the limit exists.

We try to show the limit does not exist by evaluating the limit along two different but familiar paths in the xy-plane:

ⓐ $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$

PATH along x-axis
$(x\neq 0, y=0)$

$= \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \lim_{(x,y)\to(0,0)} 0 = 0$

PATH along x-axis
$(x\neq 0, y=0)$

PATH along x-axis
$(x\neq 0, y=0)$

(Problem 32 continued)

(b) $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$$

PATH along diagonal
($y=x$, $x\neq 0$, $y\neq 0$)

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{y^2}}{3\cancel{y^2}} = \lim_{(x,y)\to(0,0)} \frac{1}{3} = \frac{1}{3}$$

PATH along diagonal
($y=x$, $x\neq 0$, $y\neq 0$)

PATH along diagonal
($y=x$, $x\neq 0$, $y\neq 0$)

Since

$$0 \neq \frac{1}{3}$$

our limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2} \quad \text{DNE}.$$

We do not bother to check the third criterion since $f(x,y)$ has already failed one of the three criteria, and it must satisfy ALL three criteria to be continuous at $(x,y)=(0,0)$.

So, $f(x,y)$ is not continuous at $(x,y)=(0,0)$.

(Problem 32 continued)

Therefore, we still end up saying

$$\boxed{\begin{array}{l} f(x,y) \text{ is continuous on} \\ \{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0)\} \end{array}}$$

(and this is the largest set that $f(x,y)$ is continuous on!).

SECTION 11.3

14. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4$

$$f_x = \frac{\partial}{\partial x}(x^5 + 3x^3y^2 + 3xy^4)$$

$\boxed{\begin{array}{l} x = \text{VARIABLE} \\ y = \text{CONSTANT} \end{array}}$

$$= \frac{\partial}{\partial x}(x^5) + 3y^2\frac{\partial}{\partial x}(x^3) + 3y^4\frac{\partial}{\partial x}(x)$$

$$= 5x^4 + 3y^2(3x^2) + 3y^4(1)$$

$$= \boxed{5x^4 + 9x^2y^2 + 3y^4}$$

$$f_y = \frac{\partial}{\partial y}(x^5) + 3x^3\frac{\partial}{\partial y}(y^2) + 3x\frac{\partial}{\partial y}(y^4)$$

$\boxed{\begin{array}{l} y = \text{VARIABLE} \\ x = \text{CONSTANT} \end{array}}$

$$= 0 + 3x^3(2y) + 3x(4y^3)$$

$$= \boxed{6x^3y + 12xy^3}$$

16. $z = y \ln x$

$$z_x = \frac{\partial}{\partial x}(y \ln x) = y \frac{\partial}{\partial x}(\ln x) = y \cdot \frac{1}{x} = \boxed{\frac{y}{x}}$$

$\boxed{\begin{array}{l} x = \text{VARIABLE} \\ y = \text{CONSTANT} \end{array}}$

$$z_y = \frac{\partial}{\partial y}(y \ln x) = \ln x \frac{\partial}{\partial y}(y) = \ln x \cdot 1 = \boxed{\ln x}$$

$\boxed{\begin{array}{l} y = \text{VARIABLE} \\ x = \text{CONSTANT} \end{array}}$

18. $f(x,y) = x^y$

$f_x = \frac{\partial}{\partial x}(x^y) = \boxed{yx^{y-1}}$

$\boxed{\begin{array}{l} x = \text{VARIABLE} \\ y = \text{CONSTANT} \end{array}}$

$\boxed{\text{LIKE } \frac{d}{dx}(x^n) = nx^{n-1}}$

$f_y = \frac{\partial}{\partial y}(x^y) = \boxed{x^y \ln x} \quad (x > 0)$

$\boxed{\begin{array}{l} y = \text{VARIABLE} \\ x = \text{CONSTANT} \end{array}}$

$\boxed{\text{LIKE } \frac{d}{dy}(a^y) = a^y \ln a}$

20. $f(s,t) = \dfrac{st^2}{s^2+t^2}$

$$f_s = \frac{\partial}{\partial s}\left(\frac{st^2}{s^2+t^2}\right)$$

S = VARIABLE
t = CONSTANT

$$= \frac{\left[\frac{\partial}{\partial s}(st^2)\right](s^2+t^2) - st^2\left[\frac{\partial}{\partial s}(s^2+t^2)\right]}{(s^2+t^2)^2}$$

$$= \frac{\left[t^2\frac{\partial}{\partial s}(s)\right](s^2+t^2) - st^2\left[2s+0\right]}{(s^2+t^2)^2}$$

$$= \frac{t^2(1)(s^2+t^2) - 2s^2t^2}{(s^2+t^2)^2}$$

$$= \frac{s^2t^2+t^4-2s^2t^2}{(s^2+t^2)^2}$$

$$= \boxed{\frac{t^4-s^2t^2}{(s^2+t^2)^2}} = \boxed{\frac{t^2(t^2-s^2)}{(s^2+t^2)^2}}$$

(Problem 20 continued)

$$f_t = \frac{\partial}{\partial t}\left(\frac{st^2}{s^2+t^2}\right)$$

$\boxed{\begin{array}{l} t = \text{VARIABLE} \\ s = \text{CONSTANT} \end{array}}$

$$= \frac{\left[\frac{\partial}{\partial t}(st^2)\right](s^2+t^2) - st^2\left[\frac{\partial}{\partial t}(s^2+t^2)\right]}{(s^2+t^2)^2}$$

$$= \frac{\left[s\frac{\partial}{\partial t}(t^2)\right](s^2+t^2) - st^2(0+2t)}{(s^2+t^2)^2}$$

$$= \frac{s(2t)(s^2+t^2) - 2st^3}{(s^2+t^2)^2}$$

$$= \frac{2s^3t + \cancel{2st^3} - \cancel{2st^3}}{(s^2+t^2)^2}$$

$$= \boxed{\frac{2s^3t}{(s^2+t^2)^2}}$$

22. $f(x,t) = e^{\sin\left(\frac{t}{x}\right)}$

$$f_x = \frac{\partial}{\partial x}\left(e^{\sin\left(\frac{t}{x}\right)}\right) = e^{\sin\left(\frac{t}{x}\right)} \cdot \frac{\partial}{\partial x}\left(\sin\left(\frac{t}{x}\right)\right)$$

X = VARIABLE

t = CONSTANT

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot \frac{\partial}{\partial x}\left(\frac{t}{x}\right)$$

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot t\frac{\partial}{\partial x}\left(\frac{1}{x}\right)$$

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot t\frac{\partial}{\partial x}\left(x^{-1}\right)$$

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot t\left(-x^{-2}\right)$$

$$= \boxed{-\frac{t}{x^2}\cos\left(\frac{t}{x}\right) e^{\sin\left(\frac{t}{x}\right)}}$$

(Problem [redacted] 22 continued)

$$f_t = \frac{\partial}{\partial t}\left(e^{\sin\left(\frac{t}{x}\right)}\right) = e^{\sin\left(\frac{t}{x}\right)} \cdot \frac{\partial}{\partial t}\left(\sin\left(\frac{t}{x}\right)\right)$$

t = VARIABLE
X = CONSTANT

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot \frac{\partial}{\partial t}\left(\frac{t}{x}\right)$$

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot \frac{1}{x}\frac{\partial}{\partial t}(t)$$

$$= e^{\sin\left(\frac{t}{x}\right)} \cdot \cos\left(\frac{t}{x}\right) \cdot \frac{1}{x}(1)$$

$$= \boxed{\frac{1}{x}\cos\left(\frac{t}{x}\right) e^{\sin\left(\frac{t}{x}\right)}}$$

26. $f(x, y, z) = x^2 e^{yz}$

$$f_x = \frac{\partial}{\partial x}\left(x^2 e^{yz}\right) = e^{yz}\frac{\partial}{\partial x}\left(x^2\right)$$

X = VARIABLE

Y, Z = CONSTANTS

$$= e^{yz}(2x) = \boxed{2xe^{yz}}$$

$$f_y = \frac{\partial}{\partial y}\left(x^2 e^{yz}\right) = x^2 \frac{\partial}{\partial y}\left(e^{yz}\right)$$

Y = VARIABLE

X, Z = CONSTANTS

$$= x^2 e^{yz} \cdot \frac{\partial}{\partial y}(yz) = x^2 e^{yz} \cdot z \frac{\partial}{\partial y}(y)$$

$$= x^2 e^{yz} \cdot z(1) = \boxed{x^2 z e^{yz}}$$

(Problem 26 continued)

$$f_z = \frac{\partial}{\partial z}\left(x^2 e^{yz}\right) = x^2 \frac{\partial}{\partial z}\left(e^{yz}\right)$$

$$\boxed{\begin{array}{l} z = \text{VARIABLE} \\ x, y = \text{CONSTANTS} \end{array}}$$

$$= x^2 e^{yz} \cdot \frac{\partial}{\partial z}(yz) = x^2 e^{yz} \cdot y \frac{\partial}{\partial z}(z)$$

$$= x^2 e^{yz} \cdot y(1) = \boxed{x^2 y e^{yz}}$$

28. $w = \sqrt{r^2 + s^2 + t^2}$

$= (r^2 + s^2 + t^2)^{1/2}$

$w_r = \frac{\partial}{\partial r}(r^2 + s^2 + t^2)^{1/2}$

$\boxed{\begin{array}{l} r = \text{VARIABLE} \\ s, t = \text{CONSTANTS} \end{array}}$

$= \frac{1}{2}(r^2 + s^2 + t^2)^{-1/2} \cdot \frac{\partial}{\partial r}(r^2 + s^2 + t^2)$

$= \frac{1}{\cancel{2}}(r^2 + s^2 + t^2)^{-1/2} \cdot (\cancel{2}r + 0 + 0)$

$= \boxed{\frac{r}{\sqrt{r^2 + s^2 + t^2}}}$

$w_s = \frac{\partial}{\partial s}(r^2 + s^2 + t^2)^{1/2}$

$\boxed{\begin{array}{l} s = \text{VARIABLE} \\ r, t = \text{CONSTANTS} \end{array}}$

$= \frac{1}{2}(r^2 + s^2 + t^2)^{-1/2} \cdot \frac{\partial}{\partial s}(r^2 + s^2 + t^2)$

$= \frac{1}{\cancel{2}}(r^2 + s^2 + t^2)^{-1/2} \cdot (0 + \cancel{2}s + 0)$

$= \boxed{\frac{s}{\sqrt{r^2 + s^2 + t^2}}}$

(Problem 28 continued)

$$\omega_t = \frac{\partial}{\partial t}(r^2+s^2+t^2)^{1/2}$$

$\boxed{\begin{array}{l} t = \text{VARIABLE} \\ r, s = \text{CONSTANTS} \end{array}}$

$$= \frac{1}{2}(r^2+s^2+t^2)^{-1/2} \cdot \frac{\partial}{\partial t}(r^2+s^2+t^2)$$

$$= \frac{1}{\cancel{2}}(r^2+s^2+t^2)^{-1/2} \cdot (0+0+\cancel{2}t)$$

$$= \boxed{\frac{t}{\sqrt{r^2+s^2+t^2}}}$$

30. $u = x^{\frac{y}{z}}$

$$u_x = \frac{\partial}{\partial x}\left(x^{\frac{y}{z}}\right) = \boxed{\frac{y}{z} \cdot x^{\frac{y}{z}-1}}$$

$\boxed{\begin{array}{l} x = \text{VARIABLE} \\ y, z = \text{CONSTANTS} \end{array}}$

$\boxed{\text{LIKE } \frac{d}{dx}(x^n) = n x^{n-1}}$

$$u_y = \frac{\partial}{\partial y}\left(x^{\frac{y}{z}}\right) = x^{\frac{y}{z}} \cdot \ln x \cdot \frac{\partial}{\partial y}\left(\frac{y}{z}\right)$$

$\boxed{\begin{array}{l} y = \text{VARIABLE} \\ x, z = \text{CONSTANTS} \end{array}}$

$\boxed{\text{LIKE } \frac{d}{dy}\left(a^{g(y)}\right) = a^{g(y)} \ln a \cdot g'(y)}$

$$= x^{\frac{y}{z}} \cdot \ln x \cdot \frac{1}{z} \frac{\partial}{\partial y}(y) = x^{\frac{y}{z}} \cdot \ln x \cdot \frac{1}{z}(1)$$

$$= \boxed{\frac{x^{\frac{y}{z}}}{z} \ln x}$$

(Problem 30 continued)

$$u_z = \frac{\partial}{\partial z}\left(x^{y/z}\right) = x^{y/z} \cdot \ln x \cdot \frac{\partial}{\partial z}\left(\frac{y}{z}\right)$$

$\boxed{\begin{array}{l} z = \text{VARIABLE} \\ x, y = \text{CONSTANTS} \end{array}}$ $\boxed{\text{LIKE } \frac{d}{dz} a^{g(z)} = a^{g(z)} \ln a \cdot g'(z)}$

$$= x^{y/z} \cdot \ln x \cdot y \frac{\partial}{\partial z}\left(\frac{1}{z}\right) = x^{y/z} \cdot \ln x \cdot y \frac{\partial}{\partial z}\left(z^{-1}\right)$$

$$= x^{y/z} \cdot \ln x \cdot y\left(-z^{-2}\right) = \boxed{-\frac{x^{y/z}\, y}{z^2} \ln x}$$

32. $f(x, y, z, t) = xy^2z^3t^4$

$$f_x = \frac{\partial}{\partial x}(xy^2z^3t^4) = y^2z^3t^4\frac{\partial}{\partial x}(x)$$

X = VARIABLE
Y, Z, t = CONSTANTS

$$= y^2z^3t^4(1) = \boxed{y^2z^3t^4}$$

$$f_y = \frac{\partial}{\partial y}(xy^2z^3t^4) = xz^3t^4\frac{\partial}{\partial y}(y^2)$$

Y = VARIABLE
X, Z, t = CONSTANTS

$$= xz^3t^4(2y) = \boxed{2xyz^3t^4}$$

$$f_z = \frac{\partial}{\partial z}(xy^2z^3t^4) = xy^2t^4\frac{\partial}{\partial z}(z^3)$$

Z = VARIABLE
X, Y, t = CONSTANTS

$$= xy^2t^4(3z^2) = \boxed{3xy^2z^2t^4}$$

(Problem 32 continued)

$$f_t = \frac{\partial}{\partial t}\left(xy^2z^3t^4\right) = xy^2z^3\frac{\partial}{\partial t}(t^4)$$

$\boxed{\begin{array}{l} t = \text{VARIABLE} \\ x, y, z = \text{CONSTANTS} \end{array}}$

$$= xy^2z^3(4t^3) = \boxed{4xy^2z^3t^3}$$

36. $f(x,y) = \sin(2x+3y)$;
$f_y(-6,4)$

$f_y(x,y) = \frac{\partial}{\partial y}\left(\sin(2x+3y)\right)$

y = VARIABLE
x = CONSTANT

$= \cos(2x+3y) \cdot \frac{\partial}{\partial y}(2x+3y)$

$= \cos(2x+3y) \cdot (0+3)$

$= \boxed{3\cos(2x+3y)}$

$\therefore f_y(-6,4) = 3\cos\left(2(-6)+3(4)\right)$

$= 3\cos(-12+12)$

$= 3\cos 0$

$= 3 \cdot 1$

$= \boxed{3}$

40. $f(x,y) = \sqrt{3x-y}$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-y} - \sqrt{3x-y}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)-y} - \sqrt{3x-y}}{h} \cdot \frac{\sqrt{3(x+h)-y} + \sqrt{3x-y}}{\sqrt{3(x+h)-y} + \sqrt{3x-y}}$$

RECALL: $(a-b)(a+b) = a^2 - b^2$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3(x+h)^2-y}\right)^2 - \left(\sqrt{3x-y}\right)^2}{h\left(\sqrt{3(x+h)-y} + \sqrt{3x-y}\right)}$$

$$= \lim_{h\to 0} \frac{[3(x+h)-y] - (3x-y)}{h\left(\sqrt{3(x+h)-y} + \sqrt{3x-y}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{3x} + 3h - \cancel{y} - \cancel{3x} + \cancel{y}}{h\left(\sqrt{3(x+h)-y} + \sqrt{3x-y}\right)}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3(x+h)-y} + \sqrt{3x-y}\right)}$$

(Problem 40 continued)

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)-y} + \sqrt{3x-y}}$$

$$= \frac{3}{\sqrt{3(x+0)-y} + \sqrt{3x-y}} = \boxed{\frac{3}{2\sqrt{3x-y}}}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x-(y+h)} - \sqrt{3x-y}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x-(y+h)} - \sqrt{3x-y}}{h} \cdot \frac{\sqrt{3x-(y+h)} + \sqrt{3x-y}}{\sqrt{3x-(y+h)} + \sqrt{3x-y}}$$

RECALL AGAIN:

$$(a-b)(a+b) = a^2 - b^2$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3x-(y+h)}\right)^2 - \left(\sqrt{3x-y}\right)^2}{h\left(\sqrt{3x-(y+h)} + \sqrt{3x-y}\right)}$$

(Problem 40 continued)

$$\stackrel{*}{=} \lim_{h \to 0} \frac{[3x-(y+h)]-(3x-y)}{h\left(\sqrt{3x-(y+h)}+\sqrt{3x-y}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{3x}-\cancel{y}-h-\cancel{3x}+\cancel{y}}{h\left(\sqrt{3x-(y+h)}+\sqrt{3x-y}\right)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\left(\sqrt{3x-(y+h)}+\sqrt{3x-y}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}}$$

$$= \frac{-1}{\sqrt{3x-(y+0)}+\sqrt{3x-y}} = \boxed{-\frac{1}{2\sqrt{3x-y}}}$$

48. $f(x,y) = \ln(3x+5y)$

FIRST PARTIALS ARE f_x, f_y

SECOND PARTIALS ARE f_{xx}, f_{yy}, f_{xy}, f_{yx}

$$f_x = \frac{\partial}{\partial x}\left(\ln(3x+5y)\right) = \frac{1}{3x+5y}\cdot\frac{\partial}{\partial x}(3x+5y)$$

$$= \frac{1}{3x+5y}\cdot(3+0) = \boxed{\frac{3}{3x+5y}}$$

$$f_y = \frac{\partial}{\partial y}\left(\ln(3x+5y)\right) = \frac{1}{3x+5y}\cdot\frac{\partial}{\partial y}(3x+5y)$$

$$= \frac{1}{3x+5y}\cdot(0+5) = \boxed{\frac{5}{3x+5y}}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(f_x)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{3x+5y}\right)$$

$$= \frac{\left[\frac{\partial}{\partial x}(3)\right](3x+5y) - 3\frac{\partial}{\partial x}(3x+5y)}{(3x+5y)^2}$$

$$= \frac{0\cdot(3x+5y) - 3(3+0)}{(3x+5y)^2} = \boxed{-\frac{9}{(3x+5y)^2}}$$

(Problem 48 continued)

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(f_y\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{5}{3x+5y}\right)$$

$$= \frac{\left[\frac{\partial}{\partial y}(5)\right](3x+5y) - 5\frac{\partial}{\partial y}(3x+5y)}{(3x+5y)^2}$$

$$= \frac{0\cdot(3x+5y) - 5(0+5)}{(3x+5y)^2} = \boxed{-\frac{25}{(3x+5y)^2}}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(f_x\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{3}{3x+5y}\right)$$

$$= \frac{\left[\frac{\partial}{\partial y}(3)\right](3x+5y) - 3\frac{\partial}{\partial y}(3x+5y)}{(3x+5y)^2}$$

$$= \frac{0\cdot(3x+5y) - 3(0+5)}{(3x+5y)^2} = \boxed{-\frac{15}{(3x+5y)^2}}$$

(Problem 48 continued)

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(f_y\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{5}{3x+5y}\right)$$

$$= \frac{\left[\frac{\partial}{\partial x}(5)\right](3x+5y) - 5\frac{\partial}{\partial x}(3x+5y)}{(3x+5y)^2}$$

$$= \frac{0\cdot(3x+5y) - 5(3+0)}{(3x+5y)^2} = \boxed{-\frac{15}{(3x+5y)^2}}$$

Observe that $f_{xy} = f_{yx}$, which should be the case since f_{xy} and f_{yx} are both continuous where f is defined (f is defined on the set $\{(x,y) \in \mathbb{R}^2 \mid 3x+5y > 0\}$). SEE Clairaut's theorem on p. 773 of your text.

50. $z = y\tan(2x)$

FIRST PARTIALS ARE z_x, z_y

SECOND PARTIALS ARE $z_{xx}, z_{yy}, z_{xy}, z_{yx}$

$$z_x = \frac{\partial}{\partial x}\left(y\tan(2x)\right) = y\frac{\partial}{\partial x}\left(\tan(2x)\right)$$

$$= y\sec^2(2x)\cdot\frac{\partial}{\partial x}(2x) = y\sec^2(2x)\cdot 2$$

$$= \boxed{2y\sec^2(2x)}$$

$$z_y = \frac{\partial}{\partial y}\left(y\tan(2x)\right) = \tan(2x)\frac{\partial}{\partial y}(y)$$

$$= \tan(2x)\cdot 1 = \boxed{\tan(2x)}$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(z_x)$$

$$= \frac{\partial}{\partial x}\left(2y\sec^2(2x)\right) = 2y\frac{\partial}{\partial x}\left[\sec(2x)\right]^2$$

$$= 2y\cdot 2\sec(2x)\cdot\frac{\partial}{\partial x}\left(\sec(2x)\right)$$

FROM Item # 17, REFERENCE PAGE 3, BACK OF YOUR TEXT: $\frac{d}{dx}(\sec x) = \sec x\tan x$

(Problem 50 continued)

$$= 2y \cdot 2\sec(2x) \cdot \sec(2x)\tan(2x) \cdot \frac{\partial}{\partial x}(2x)$$

$$= 2y \cdot 2\sec(2x) \cdot \sec(2x)\tan(2x) \cdot 2$$

$$= \boxed{8y\sec^2(2x)\tan(2x)}$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(z_y)$$

$$= \frac{\partial}{\partial y}\left(\tan(2x)\right) = \boxed{0}$$

$$z_{xy} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(z_x)$$

$$= \frac{\partial}{\partial y}\left(2y\sec^2(2x)\right) = 2\sec^2(2x)\frac{\partial}{\partial y}(y)$$

$$= 2\sec^2(2x) \cdot 1 = \boxed{2\sec^2(2x)}$$

(Problem 50 continued)

$$z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(z_y)$$

$$= \frac{\partial}{\partial x}\left(\tan(2x)\right)$$

$$= \sec^2(2x) \cdot \frac{\partial}{\partial x}(2x)$$

$$= \sec^2(2x) \cdot 2 = \boxed{2\sec^2(2x)}$$

Again, observe that $f_{xy} = f_{yx}$, which should be the case since f_{xy} and f_{yx} are both continuous where f is defined (f is defined on the set $\{(x,y) \in \mathbb{R}^2 \mid \cos 2x \neq 0\}$).

SEE Clairaut's theorem on p. 773 of your text.

52. $u = xye^y$

Essentially

Clairaut's theorem: $u_{xy} = u_{yx}$.

$$u_x = \frac{\partial}{\partial x}(xye^y) = ye^y \frac{\partial}{\partial x}(x) = ye^y \cdot 1$$
$$= \boxed{ye^y}$$

$$u_y = \frac{\partial}{\partial y}(xye^y) = x\frac{\partial}{\partial y}(ye^y)$$
$$= x\left[\left(\frac{\partial}{\partial y}y\right)e^y + y\frac{\partial}{\partial y}(e^y)\right]$$
$$= x(1\cdot e^y + y\cdot e^y) = \boxed{x(1+y)e^y}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(u_x)$$
$$= \frac{\partial}{\partial y}(ye^y) = \left(\frac{\partial}{\partial y}y\right)e^y + y\frac{\partial}{\partial y}(e^y)$$
$$= 1\cdot e^y + y\cdot e^y = \boxed{(1+y)e^y}$$

(Problem 52 continued)

$$u_{yx} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(u_y)$$

$$= \frac{\partial}{\partial x}\left(x(1+y)e^y\right) = (1+y)e^y \frac{\partial}{\partial x}(x)$$

$$= (1+y)e^y \cdot 1 = \boxed{(1+y)e^y}$$

$$\therefore \quad u_{xy} = u_{yx} \quad \checkmark$$

This should be expected since u_{xy} and u_{yx} are continuous everywhere in $\mathbb{R}^2$ and $u = xye^y$ is defined everywhere in $\mathbb{R}^2$.

54. $f(x,y) = e^{xy^2}$; f_{xxy}

$$f_x = \frac{\partial}{\partial x}\left(e^{xy^2}\right) = e^{xy^2} \cdot \frac{\partial}{\partial x}(xy^2)$$

$$= e^{xy^2} \cdot y^2 \frac{\partial}{\partial x}(x) = e^{xy^2} \cdot y^2 \cdot 1$$

$$= \boxed{y^2 e^{xy^2}}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f_x\right)$$

$$= \frac{\partial}{\partial x}\left(y^2 e^{xy^2}\right) = y^2 \frac{\partial}{\partial x}\left(e^{xy^2}\right)$$

$$= y^2 \cdot e^{xy^2} \cdot \frac{\partial}{\partial x}(xy^2) =$$

$$= y^2 \cdot e^{xy^2} \cdot y^2 \frac{\partial}{\partial x}(x)$$

$$= y^2 \cdot e^{xy^2} \cdot y^2 \cdot 1 = \boxed{y^4 e^{xy^2}}$$

(Problem 54 continued)

$$f_{xxy} = \frac{\partial^3 f}{\partial y\, \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(f_{xx}\right)$$

$$= \frac{\partial}{\partial y}\left(y^4 e^{xy^2}\right)$$

$$= \left[\frac{\partial}{\partial y}(y^4)\right] e^{xy^2} + y^4 \frac{\partial}{\partial y}\left(e^{xy^2}\right)$$

$$= 4y^3 e^{xy^2} + y^4 \cdot e^{xy^2} \cdot \frac{\partial}{\partial y}(xy^2)$$

$$= 4y^3 e^{xy^2} + y^4 \cdot e^{xy^2} \cdot x \frac{\partial}{\partial y}(y^2)$$

$$= 4y^3 e^{xy^2} + y^4 \cdot e^{xy^2} \cdot x(2y)$$

$$= \boxed{4y^3 e^{xy^2} + 2xy^5 e^{xy^2}}$$

$$= \boxed{2y^3 e^{xy^2}(2 + xy^2)}$$

56. $f(x,y,z) = e^{xyz}$; f_{yzy}

$$f_y = \frac{\partial}{\partial y}\left(e^{xyz}\right) = e^{xyz} \cdot \frac{\partial}{\partial y}(xyz)$$

$$= e^{xyz} \cdot xz \frac{\partial}{\partial y}(y) = e^{xyz} \cdot xz(1)$$

$$= \boxed{xz e^{xyz}}$$

1

$$f_{yz} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}(f_y)$$

$$= \frac{\partial}{\partial z}\left(xz e^{xyz}\right) = x \frac{\partial}{\partial z}\left(z e^{xyz}\right)$$

$$= x\left[\left(\frac{\partial}{\partial z} z\right) e^{xyz} + z \frac{\partial}{\partial z}\left(e^{xyz}\right)\right]$$

$$= x\left[1 \cdot e^{xyz} + z \cdot e^{xyz} \cdot \frac{\partial}{\partial z}(xyz)\right]$$

$$= x\left[1 \cdot e^{xyz} + z \cdot e^{xyz} \cdot xy \frac{\partial}{\partial z}(z)\right]$$

$$= x\left[1 \cdot e^{xyz} + z \cdot e^{xyz} \cdot xy(1)\right]$$

$\downarrow$

(Problem 56 continued)

$$= x\left(e^{xyz} + xyz\, e^{xyz}\right)$$

$$= \boxed{x e^{xyz}(1 + xyz)}$$

$$f_{yzy} = \frac{\partial^3 f}{\partial y\, \partial z\, \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial z\, \partial y}\right) = \frac{\partial}{\partial y}\left(f_{yz}\right)$$

$$= \frac{\partial}{\partial y}\left(x e^{xyz}(1 + xyz)\right)$$

$$= x \frac{\partial}{\partial y}\left(e^{xyz}(1+xyz)\right)$$

$$= x\left[\left(\frac{\partial}{\partial y} e^{xyz}\right)(1+xyz) + e^{xyz}\frac{\partial}{\partial y}(1+xyz)\right]$$

$$= x\left[e^{xyz}\left(\frac{\partial}{\partial y} xyz\right)(1+xyz) + e^{xyz}\left(0 + \frac{\partial}{\partial y}(xyz)\right)\right]$$

$$= x\left[e^{xyz} \cdot xz \cdot \left(\frac{\partial}{\partial y} y\right)(1+xyz) + e^{xyz} \cdot xz \frac{\partial}{\partial y}(y)\right]$$

(Problem 56 continued)

$$= x\left[e^{xyz}\cdot xz(1)(1+xyz)+e^{xyz}\cdot xz\,(1)\right]$$

$$= \boxed{x\left[xz(1+xyz)e^{xyz}+xz\,e^{xyz}\right]}$$

$$= x^2 z\, e^{xyz}(1+xyz+1)$$

$$= \boxed{x^2 z\, e^{xyz}\,(2+xyz)}$$