Lecture

Section 11.6. Directional Derivatives and the Gradient Vector

Consider the real-valued function of a single variable:

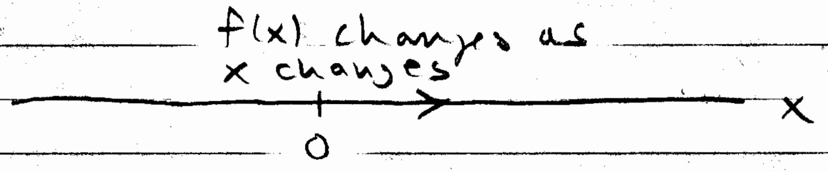
$$f(x).$$

Then its derivative

$$\frac{df}{dx}$$

tells us how f is changing as x is changing, or,

HOW f IS CHANGING
ALONG THE DIRECTION
OF THE x-AXIS



Now consider the real-valued function of two variables:

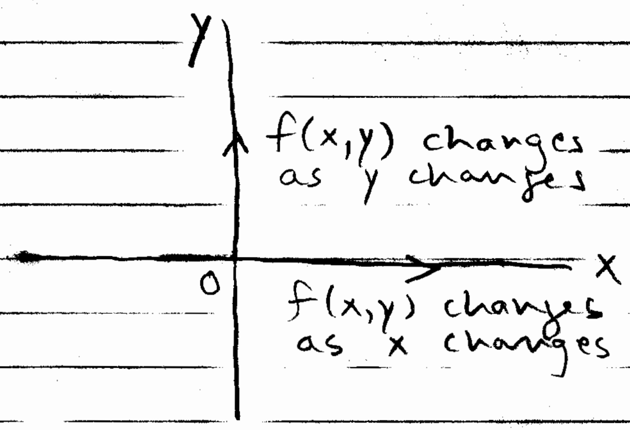
$$f(x, y)$$

Then its partial derivatives

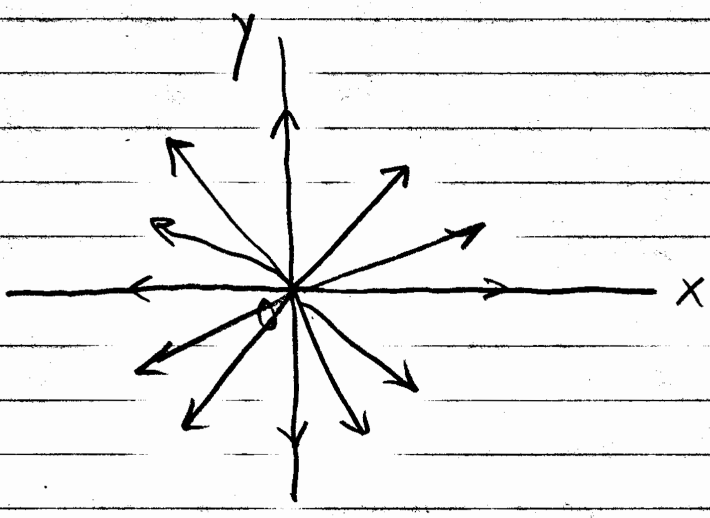
$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

tell us how f is changing as x is changing or as y is changing, or,

HOW f IS CHANGING ALONG THE DIRECTION OF THE x-AXIS OR THE y-AXIS

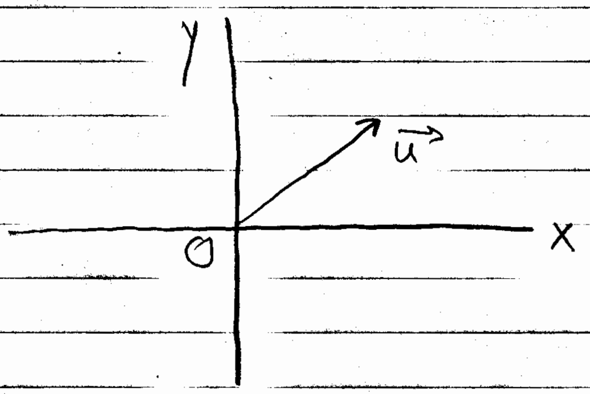


BUT there are infinitely many other directions in the xy-plane over which $f(x, y)$ can change!



These directions can be represented by unit vectors

$$\vec{u} = \langle a, b \rangle, \quad |\vec{u}| = 1,$$



Defn. The DIRECTIONAL DERIVATIVE of $f(x,y)$ the point (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y + hb) - f(x_0, y_0)}{h}}$$

<u>Working Defn.</u> The DIRECTIONAL DERIVATIVE of $f(x,y)$ in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is

$$\boxed{\begin{aligned} D_{\vec{u}} f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\ &= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle \end{aligned}}$$

Specifically, the DIRECTIONAL DERIVATIVE of f at the point (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is

$$\boxed{D_{\vec{u}} f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0) \rangle \cdot \langle a, b \rangle}$$

Text's Proof :

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{\overbrace{f(x_0+ha, y_0+hb)}^{\equiv g(h)} - \overbrace{f(x_0, y_0)}^{= g(0)}}{h}$$

$$= \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$\equiv g'(h)$$

$$= \frac{dg}{dh}$$

$$\equiv \frac{d}{dh}\Big(f(\overbrace{x_0+ha}^{\equiv x}, \overbrace{y_0+hb}^{\equiv y})\Big)$$

CHAIN RULE from Sect. 11.5

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh}$$

$$= f_x(x, y) \cdot \frac{d}{dh}(x_0 + ha)$$

$$+ f_y(x, y) \cdot \frac{d}{dh}(y_0 + hb)$$

$$= f_x(x,y)\cdot a + f_y(x,y)\cdot b$$
$$= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle \quad \blacksquare$$

---o---

We next give the vector
$$\langle f_x(x,y), f_y(x,y)\rangle$$
a NAME and DEFINITION, because this vector keeps coming up in the following sections and chapters.

Defn. Let $f(x,y)$ be a function of two variables. Then the GRADIENT of f is denoted by

$$\nabla f \quad \text{or} \quad \nabla f(x,y)$$

and given by

$$\boxed{\nabla f(x,y) = \langle f_x(x,y),\, f_y(x,y)\rangle.}$$

Remark. The DIRECTIONAL DERIVATIVE of $f(x,y)$ can be written as

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$

or

$$\boxed{D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}}$$

where

$$\vec{x} = (x,y)$$

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Given the Definition and Remark above for GRADIENT of a function of two variables, we now can with some relative ease describe two very important results.

Result. The MAXIMUM VALUE of $D_{\vec{u}} f(\vec{x})$ over all possible directions $\vec{u}$ is

i.e., the maximum value of $D_{\vec{u}} f$ over all possible $\vec{u}$'s

$$\max_{\vec{u} \in \mathbb{R}^2} \{ D_{\vec{u}} f(\vec{x}) \} = |\nabla f(\vec{x})| ,$$

i.e., the maximum rate of change of f occurs along the gradient vector and

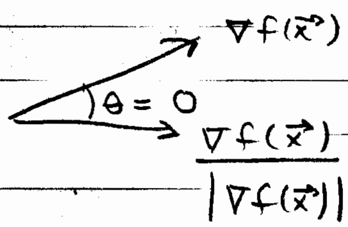
$$\vec{u} = \frac{\nabla f(\vec{x})}{|\nabla f(\vec{x})|} .$$

Proof.

$$D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$$

$$= \nabla f(\vec{x}) \cdot \frac{\nabla f(\vec{x})}{|\nabla f(\vec{x})|}$$

$$= |\nabla f(\vec{x})| \left| \frac{\nabla f(\vec{x})}{|\nabla f(\vec{x})|} \right| \cos\theta$$

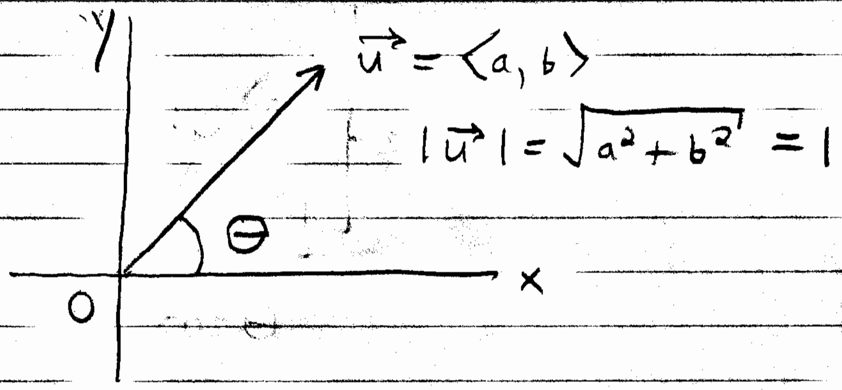


$$= |\nabla f(\vec{x})| \, |1| \cos 0$$

$$= |\nabla f(\vec{x})| \cdot 1 \cdot 1$$

$$= |\nabla f(\vec{x})| \quad \blacksquare$$

<u>Result</u>. If we are given θ such that



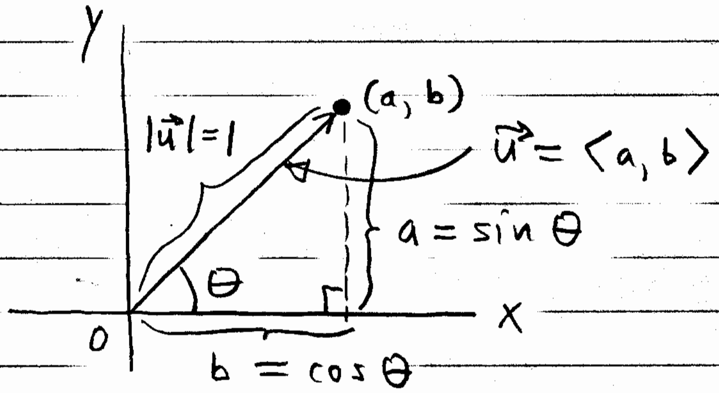
then we can write

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \langle \cos\theta, \sin\theta \rangle}$$

<u>Proof</u>,

$$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \langle a, b \rangle$$

where



$\blacksquare$

The ideas of DIRECTIONAL DERIVATIVE and its "sidekick" GRADIENT can be <u>extended to</u> functions of more than two variables:

$$\left.\begin{array}{l} z = f(\underbrace{x_1, x_2, \ldots, x_n}) \\ \quad\text{in } \mathbb{R}^n \text{ (n dimensions)} \\ \qquad\text{and} \\ \vec{u} = \langle a_1, a_2, \ldots, a_n \rangle \text{ in } \mathbb{R}^n \end{array}\right\} \Rightarrow$$

$$\nabla f(x_1, x_2, \ldots, x_n) = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

$$= \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle$$

and

$$D_{\vec{u}} f(x_1, x_2, \ldots, x_n) = \nabla f(x_1, x_2, \ldots, x_n) \cdot \vec{u}$$

$$= f_{x_1} a_1 + f_{x_2} a_2 + \cdots + f_{x_n} a_n$$

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Exercises 11.6, pp. 808-810.

4-6 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

5. $f(x,y) = \sqrt{5x-4y}$, $(4,1)$, $\theta = -\pi/6$

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$f_x = \frac{\partial}{\partial x}(5x-4y)^{1/2}$$
$$= \frac{1}{2}(5x-4y)^{-1/2} \cdot \frac{\partial}{\partial x}(5x-4y)$$
$$= \frac{1}{2}(5x-4y)^{-1/2} \cdot 5$$
$$= \boxed{\frac{5}{2\sqrt{5x-4y}}}$$

$$f_y = \frac{\partial}{\partial y}(5x-4y)^{1/2}$$

$$= \frac{1}{2}(5x-4y)^{-1/2} \cdot \frac{\partial}{\partial y}(5x-4y)$$

$$= \frac{1}{\cancel{2}}(5x-4y)^{-1/2} \cdot (-\cancel{4}^{\,2})$$

$$= \boxed{-\frac{2}{\sqrt{5x-4y}}}$$

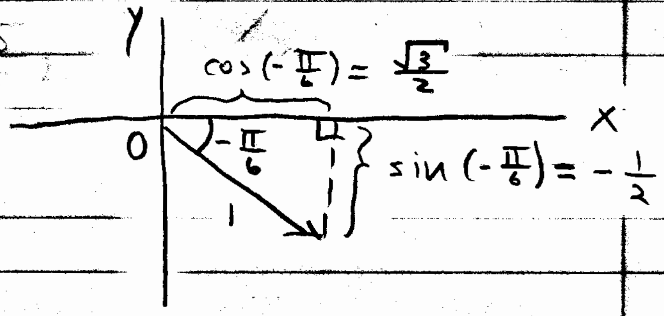
$$\therefore \nabla f(x,y) = \left\langle \frac{5}{2\sqrt{5x-4y}}, -\frac{2}{\sqrt{5x-4y}} \right\rangle$$

$$\therefore \nabla f(4,1) = \left\langle \frac{5}{2\sqrt{5(4)-4(1)}}, -\frac{2}{\sqrt{5(4)-4(1)}} \right\rangle$$

$$= \boxed{\left\langle \frac{5}{8}, -\frac{1}{2} \right\rangle}$$

So,

$$D_{\vec{u}} f(4,1) = \nabla f(4,1) \cdot \left\langle \cos\left(-\tfrac{\pi}{6}\right), \sin\left(-\tfrac{\pi}{6}\right) \right\rangle$$



$$-\tfrac{\pi}{6} = -30^\circ = 330^\circ$$

$$= \left\langle \frac{5}{8}, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

$$= \left(\frac{5}{8}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$= \boxed{\frac{5\sqrt{3}}{16} + \frac{1}{4}}$$

7-10 (a) Find the gradient of f.
(b) Evaluate the gradient at the point P.
(c) Find the rate of change of f at P in the direction of the vector $\vec{u}$.

9. $f(x,y,z) = xy^2z^3$, $P(1,-2,1)$,
$\vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$
$= \boxed{\langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle}$

(b) $\nabla f(1,-2,1) = \langle (-2)^2(1)^3, 2(1)(-2)(1)^3, 3(1)(-2)^2(1)^2 \rangle$
$= \boxed{\langle 4, -4, 12 \rangle}$

(c) $D_{\vec{u}} f(1,-2,1) = \nabla f(1,-2,1) \cdot \vec{u}$

$$= \langle 4, -4, 12 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{12}{\sqrt{3}}$$

$$= \boxed{\frac{20}{\sqrt{3}}}$$

11-15 Find the directional derivative of the function at the given point in the given direction $\vec{v}$

13. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $(1,2,-2)$,
$\vec{v} = \langle -6, 6, -3 \rangle$

Our directional derivative must be in the direction of a unit vector $\vec{u}$, where we want

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}, \quad |\vec{u}| = 1.$$

So, define

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|},$$

where

$$|\vec{u}| = \left| \frac{\vec{v}}{|\vec{v}|} \right| = \frac{|\vec{v}|}{||\vec{v}||} = \frac{|\vec{v}|}{|\vec{v}|} = 1.$$

We compute $\vec{u}$:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -6, 6, 3 \rangle}{\sqrt{(-6)^2+(6)^2+(3)^2}}$$

$$= \frac{\langle -6, 6, -3 \rangle}{9}$$

$$= \left\langle -\frac{6}{9}, \frac{6}{9}, -\frac{3}{9} \right\rangle$$

$$= \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$$

Next we compute the gradient of f :

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \left\langle \frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}, \frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2}, \frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2} \right\rangle$$

$$= \left\langle \frac{1}{\cancel{2}} \cdot (x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}x, \frac{1}{\cancel{2}} \cdot (x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}y, \frac{1}{\cancel{2}} \cdot (x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}z \right\rangle$$

$$= \boxed{\left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle}$$

Given $\vec{u}$ and ∇f we now can compute the directional derivative of f:

$$D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle \cdot \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$$

$$= \boxed{-\frac{2x}{3\sqrt{x^2+y^2+z^2}} + \frac{2y}{3\sqrt{x^2+y^2+z^2}} - \frac{z}{3\sqrt{x^2+y^2+z^2}}}$$

Finally, we compute $D_{\vec{u}} f$ at the point $(1, 2, -2)$:

$$D_{\vec{u}} f(1,2,-2) = -\frac{2(1)}{3\sqrt{(1)^2+(2)^2+(-2)^2}}$$

$$+\frac{2(2)}{3\sqrt{(1)^2+(2)^2+(-2)^2}}$$

$$-\frac{(-2)}{3\sqrt{(1)^2+(2)^2+(-2)^2}}$$

$$= -\frac{2}{3(3)} + \frac{4}{3(3)} + \frac{2}{3(3)}$$

$$= -\cancel{\frac{2}{9}} + \frac{4}{9} + \cancel{\frac{2}{9}}$$

$$= \boxed{\frac{4}{9}}$$

17. Find the directional derivative of

$$f(x,y) = \sqrt{xy}$$

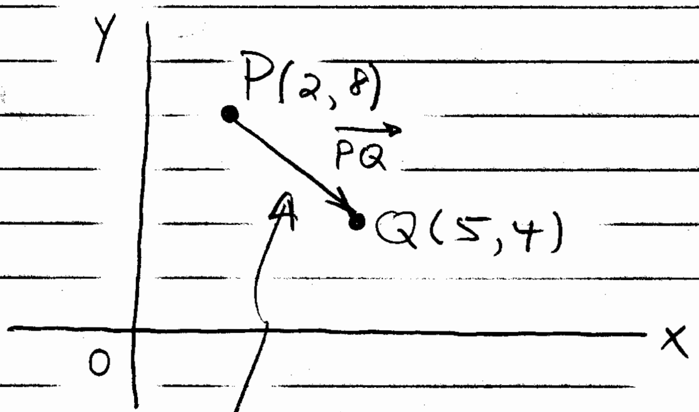
at

$$P(2,8)$$

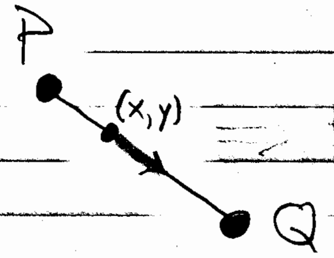
in the direction of

$$Q(5,4).$$

We wish to find how f changes in the following direction:

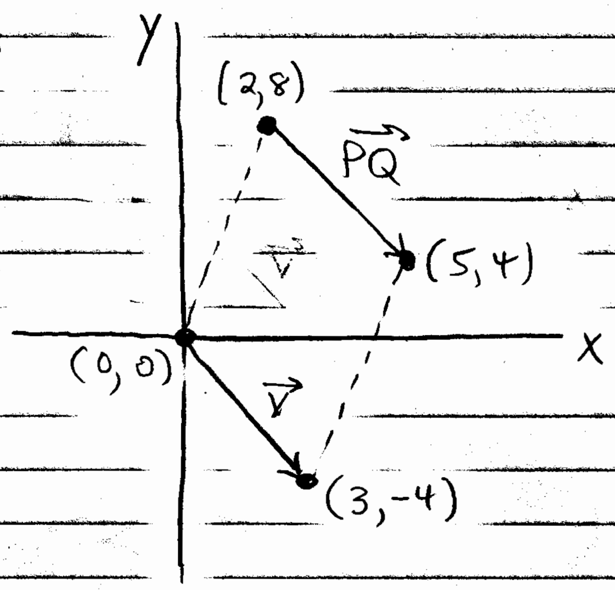


$f(x,y)$ changes as (x,y) changes, when, as as a point, (x,y) moves in the direction of $\overrightarrow{PQ}$.



In order to compute the directional derivative of f, $D_{\vec{u}} f$, we need to find what the displacement vector $\overrightarrow{PQ}$ is as a position vector:

$$\begin{aligned}\vec{v} &= \overrightarrow{PQ} \text{ as a position vector} \\ &= \text{"}Q\text{"} - \text{"}P\text{"} \\ &= \langle 5, 4 \rangle - \langle 2, 8 \rangle \\ &= \langle 3, -4 \rangle\end{aligned}$$



We then need to scale down $\vec{v}$ so that we obtain the unit vector, $\vec{u}$, with the same direction as $\vec{v}$:

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3, -4\rangle}{\sqrt{(3)^2+(-4)^2}}$$

$$= \frac{\langle 3, -4\rangle}{5}$$

$$= \boxed{\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle}$$

We now have the $\vec{u}$ in our "$D_{\vec{u}} f$." We next need ∇f:

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

VAR. CONST. CONST. VAR.

$$= \left\langle \frac{\partial}{\partial x}(xy)^{1/2}, \frac{\partial}{\partial y}(xy)^{1/2} \right\rangle$$

Recall the CHAIN RULE for $[g(x)]^n$:

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

e.g., $\frac{d}{dx}(3x)^{1/2} = \frac{1}{2}(3x)^{-1/2} \cdot 3$

$$= \left\langle \frac{1}{2}(xy)^{-1/2}\cdot\frac{\partial}{\partial x}(\underset{\text{VARIABLE}}{x}\,\underset{\text{CONSTANT}}{y}),\ \frac{1}{2}(xy)^{-1/2}\cdot\frac{\partial}{\partial y}(\underset{\text{CONSTANT}}{x}\,\underset{\text{VARIABLE}}{y}) \right\rangle$$

$$= \left\langle \frac{1}{2}\cdot\frac{1}{\sqrt{xy}}\cdot y,\ \frac{1}{2}\,\frac{1}{\sqrt{xy}}\cdot x \right\rangle$$

$$= \boxed{\left\langle \frac{y}{2\sqrt{xy}},\ \frac{x}{2\sqrt{xy}} \right\rangle}$$

We are ready to compute $D_{\vec{u}}f$:

$$D_{\vec{u}}f(x,y) = \nabla f(x,y)\cdot\vec{u}$$

$$= \left\langle \frac{y}{2\sqrt{xy}},\ \frac{x}{2\sqrt{xy}} \right\rangle\cdot\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

$$= \boxed{\frac{3y}{10\sqrt{xy}} - \frac{2x}{5\sqrt{xy}}}$$

Finally, we can compute $D_{\vec{u}} f$ at the point $P(2, 8)$:

$$D_{\vec{u}} f(2,8) = \frac{3(8)}{10\sqrt{(2)(8)}} - \frac{2(2)}{5\sqrt{(2)(8)}}$$

$$= \frac{24}{10(4)} - \frac{4}{5(4)}$$

$$= \frac{24}{40} - \frac{4}{20}$$

$$= \frac{24}{40} - \frac{8}{40}$$

$$= \frac{16}{40}$$

$$= \boxed{\frac{2}{5}}$$

19-22 Find the maximum rate of change of f at the given point and the direction in which it [the maximum rate] occurs.

19. $f(x,y) = \sin(xy)$, $(1,0)$

The maximum rate of change is given by

$$\text{MAX RATE OF CHANGE at } (0,1) = \underset{\vec{u}\in\mathbb{R}^2}{\max}\{D_{\vec{u}}f(0,1)\} = |\nabla f(0,1)|,$$

So, we first compute $\nabla f(x,y)$:

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

$$= \left\langle \frac{\partial}{\partial x}\sin(xy), \frac{\partial}{\partial y}\sin(xy) \right\rangle$$

VAR. CONST. CONST. VAR.

RECALL the formula (which is from the CHAIN RULE):

$$\frac{d}{dx}\sin(ax) = a\cos(ax)$$

e.g., $\frac{d}{dx}\sin(3x) = 3\cos(3x)$

$$= \boxed{\langle y\cos(xy),\ x\cos(xy)\rangle}$$

Next we compute $\nabla f(x,y)$ at the point $(1,0)$:

$$\nabla f(1,0) = \langle 0\cdot\cos(1\cdot 0),\ 1\cdot\cos(1\cdot 0)\rangle$$

$$= \langle 0\cdot\cos 0,\ 1\cdot\cos 0\rangle$$

$$= \langle 0\cdot 1,\ 1\cdot 1\rangle$$

$$= \boxed{\langle 0, 1\rangle}\ (=\vec{j})$$

Therefore, the maximum rate of change of f is in the direction

$$\nabla f(0,1) = \boxed{\langle 0, 1\rangle = \vec{j}}$$

and is equal to

$$|\nabla f(0,1)| = |\langle 0, 1\rangle| = \sqrt{0^2+1^2} = \boxed{1}$$

LIKE EXERCISE 24:

Find the directions in which the directional derivative of

$$f(x,y) = \ln(x^2+y^2)$$

at the point $(1,1)$ has the value 1.

First of all, we will use the formula

$$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}.$$

where we will let

$|\vec{u}|=1$

$\vec{u} = \langle a, b \rangle$ <u>such</u> <u>that</u> $\sqrt{a^2+b^2} = 1.$

We first compute ∇f:

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

$$= \left\langle \frac{\partial}{\partial x} \ln(\underset{\text{VAR.}}{x^2} + \underset{\text{CONST.}}{y^2}), \frac{\partial}{\partial y} \ln(\underset{\text{CONST.}}{x^2} + \underset{\text{VAR.}}{y^2}) \right\rangle$$

$\downarrow$

RECALL the CHAIN RULE for $\ln(g(x))$:

$$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$$

$$= \boxed{\left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle}$$

Then $\nabla f(x,y)$ at the point $(1,1)$ is

$$\nabla f(1,1) = \left\langle \frac{2(1)}{1^2+1^2}, \frac{2(1)}{1^2+1^2} \right\rangle$$

$$= \langle 1, 1 \rangle$$

Therefore, $D_{\vec{u}}f$ at the point $(1,1)$ is

$$D_{\vec{u}}f(1,1) = \nabla f(1,1) \cdot \vec{u}$$

$$= \langle 1, 1 \rangle \cdot \langle a, b \rangle$$

$$= \boxed{a + b}$$

an

Now, to find all unit vectors $\vec{u} = \langle a, b \rangle$ along which

$$D_{\vec{u}} f(1,1) = 1,$$

we must solve the following two equations in the two unknowns a and b:

$$\begin{cases} a + b = 1, \leftarrow \text{From } D_{\vec{u}} f(1,1) = a + b = 1 \\ a^2 + b^2 = 1 \leftarrow \text{From the fact that we must have } |\vec{u}| = 1 \Rightarrow |\langle a, b \rangle| = 1 \Rightarrow \sqrt{a^2+b^2} = 1 \Rightarrow \left(\sqrt{a^2+b^2}\right)^2 = 1^2 \end{cases}$$

We will "play around" with some algebra, since solving this pair of equations is not straightforward in that $a^2+b^2=1$ is nonlinear:

$$a + b = 1 \Rightarrow a = 1 - b$$
$$\Rightarrow a^2 = (1-b)^2$$
$$\Rightarrow a^2 = 1 - 2b + b^2 \quad (1)$$

$a^2+b^2=1 \Longrightarrow \boxed{a^2 = 1-b^2}$ ②

Substitute ② into ① :

$\cancel{1}-b^2 = \cancel{1}-2b+b^2 \Longrightarrow$

$0 = -2b+2b^2 \Longrightarrow$

$2b^2-2b=0 \Longrightarrow$

$b^2-b=0 \Longrightarrow$

$b(b-1)=0 \Longrightarrow$

$\boxed{b=0 \text{ or } b=1}$

$\therefore a+b=1 \Longrightarrow$ When $b=0$, $a=1$
When $b=1$, $a=0$

Therefore, there are two unit vectors, $\vec{u}=\langle a,b\rangle$ along which $D_{\vec{u}}f(1,1)=1$, and they are

$$\boxed{\vec{u}=\langle 0,1\rangle=\vec{j}, \quad \vec{u}=\langle 1,0\rangle=\vec{i}}$$

25. Find all points at which the direction of fastest change of the function

$$f(x,y) = x^2 + y^2 - 2x - 4y$$

is

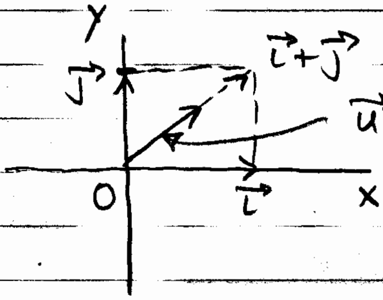
$$\vec{i} + \vec{j}$$

We need to find all points (x,y) such that

$$D_{\vec{u}} f(x,y) = |\nabla f(x,y)| \Longrightarrow$$

$$\nabla f(x,y) \cdot \vec{u} = |\nabla f(x,y)| \Longrightarrow$$

$$\nabla f(x,y) \cdot \frac{\vec{i}+\vec{j}}{|\vec{i}+\vec{j}|} = |\nabla f(x,y)| \Longrightarrow$$



$$\vec{u} = \frac{\vec{i}+\vec{j}}{|\vec{i}+\vec{j}|} = \frac{\langle 1, 1\rangle}{|\langle 1, 1\rangle|} = \frac{\langle 1, 1\rangle}{\sqrt{1^2+1^2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\boxed{\nabla f(x,y) \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = |\nabla f(x,y)|}$$

Now,

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$
$$= \langle 2x-2, 2y-4 \rangle$$

and so

$$\nabla f(x,y) \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$
$$= \langle 2x-2, 2y-4 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$
$$= \boxed{\frac{2}{\sqrt{2}}(x-1) + \frac{2}{\sqrt{2}}(y-2)}$$

and

$$|\nabla f(x,y)| = \sqrt{f_x^2 + f_y^2}$$
$$= \sqrt{(2x-2)^2 + (2y-4)^2}$$
$$= \sqrt{4(x-1)^2 + 4(y-2)^2}$$
$$= \boxed{2\sqrt{(x-1)^2 + (y-2)^2}}$$

Thus,

$$\nabla f(x,y)\cdot\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle = |\nabla f(x,y)| \Longrightarrow$$

$$\frac{\cancel{2}}{\sqrt{2}}(x-1)+\frac{\cancel{2}}{\sqrt{2}}(y-2) = \cancel{2}\sqrt{(x-1)^2+(y-2)^2} \Longrightarrow$$

$$(x-1)+(y-2) = \sqrt{2}\cdot\sqrt{(x-1)^2+(y-2)^2} \Longrightarrow$$

$$[(x-1)+(y-2)]^2 = (\sqrt{2})^2\left(\sqrt{(x-1)^2+(y-2)^2}\right)^2 \Longrightarrow$$

$$(x-1)^2+2(x-1)(y-2)+(y-2)^2 = 2[(x-1)^2+(y-2)^2] \Longrightarrow$$

$$2(x-1)(y-2) = (x-1)^2+(y-2)^2 \Longrightarrow$$

$$0 = (x-1)^2-2(x-1)(y-2)+(y-2)^2 \Longrightarrow$$

$$0 = [(x-1)-(y-2)]^2 \Longrightarrow$$

$$0 = (x-y+1)^2 \Longrightarrow$$

$$x-y+1 = 0 \Longrightarrow$$

$$\boxed{y = x+1}$$ (All po[illegible] w[illegible] on this)

OR

All points on the LINE $y = x + 1$