Lecture

4/10/01
Tues.
Lecture

Sections 4.2 and 4.3. Maximum/Minimum Values of Functions and Derivatives/Shapes of Curves.

Here we will determine what values of x will give us the largest and smallest (absolutely or relatively) values of $f(x)$.

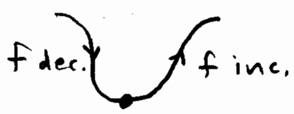
Very often this kind of information is needed in applications.

Given a function $f(x)$, we will be making use of $f'(x)$ and $f''(x)$.

We will be interested in points x where the graph of $f(x)$ changes in some way. Specifically:

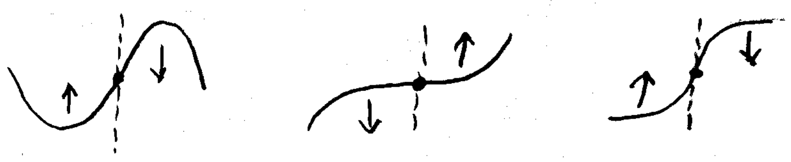
1) peaks:

2) valleys:



3) critical points: includes where peaks and valleys occur

4) <u>inflection points</u>: where concavity (bending of a curve) changes



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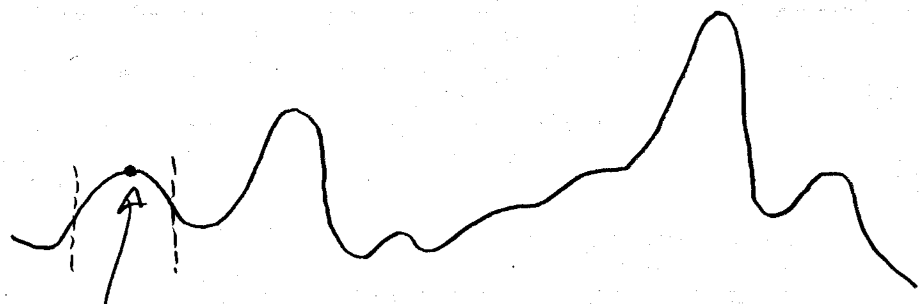
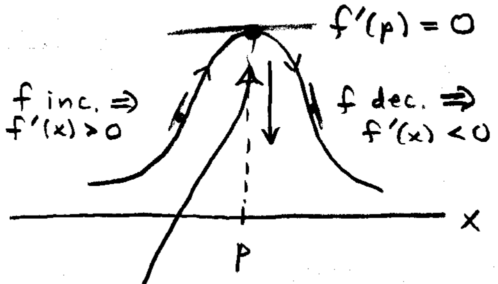
1. <u>Local Maximum</u>:
   - a peak (among other peaks)



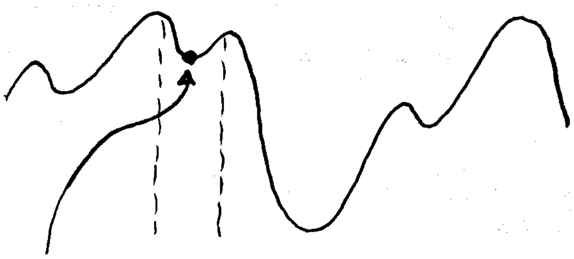
Locally, this is the highest point around.

   - with derivative $f' = 0$



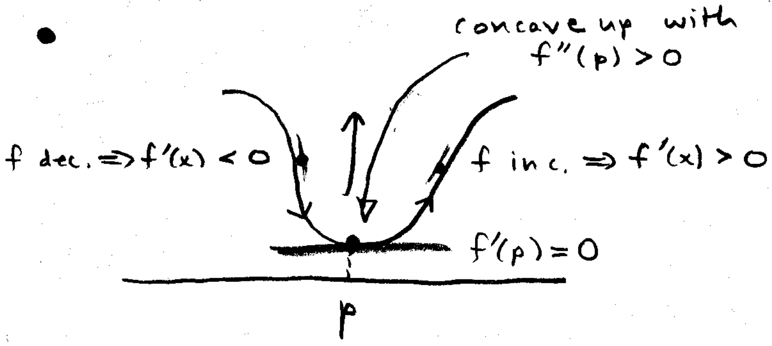
Concave down with $f''(\ ) < 0$

2. <u>Local Minimum</u>: • a valley (among other valleys)



Locally, this is the lowest point around.

•



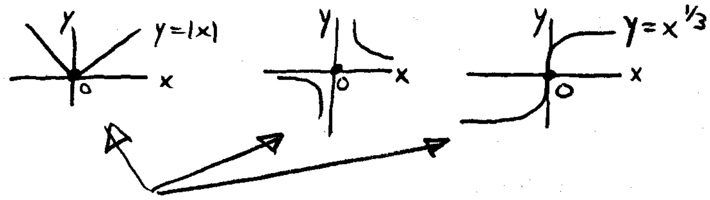
3. <u>Critical Points</u>: • <u>Defn.</u> $x = p$ is a <u>critical point</u> if either

$$f'(p) = 0$$

or

$$f'(p) = \text{undefined } (= \infty).$$

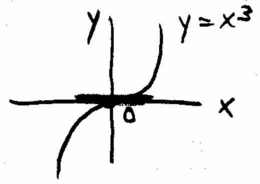
E.g.,



$f'(0)$ undefined $\Rightarrow$ $x = 0$ is a critical point

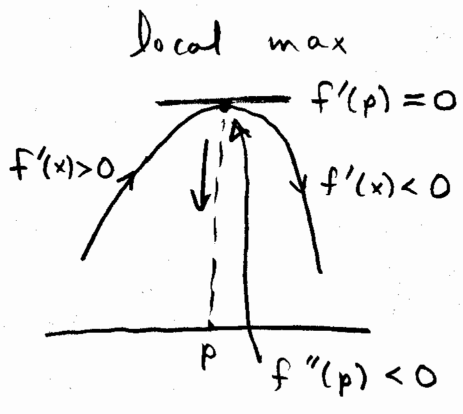
- WARNING:
  If f has a local max or min at $x=p$, then $f'(p)=0$.
  HOWEVER, if $f'(p)=0$, f may or may not have a local max or min at $x=p$!

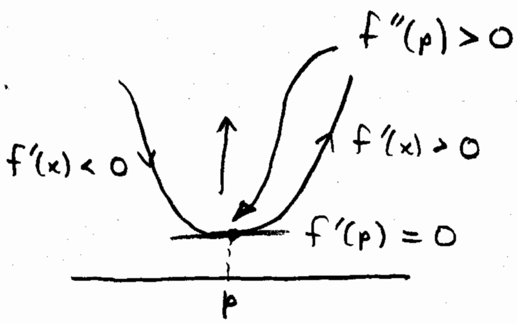
E.g.,



$f'(0)=0$ BUT there is no local max or min at $x=0$

First / Second Derivative Tests for Locating Local Maxima and Minima





FIRST DERIVATIVE TEST:

Assume $f'(p) = 0$.

1. If $f'(x) > 0$ "right before" $x = p$ and $f'(x) < 0$ "right after" $x = p$, then $f(x)$ has a LOCAL MAX at $x = p$.
2. If $f'(x) < 0$ "right before" $x = p$ and $f'(x) > 0$ "right after" $x = p$, then $f(x)$ has a LOCAL MIN at $x = p$
3. OTHERWISE, $f(x)$ DOES NOT HAVE A LOCAL MAX or MIN at $x = p$

MEMORIZE

This test will always work

This test will not always work

SECOND DERIVATIVE TEST

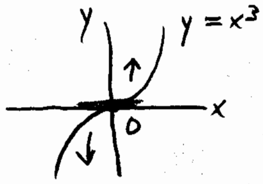
Assume $f'(p) = 0$.

1. If $f''(p) < 0$, then $f(x)$ has a LOCAL MAX at $x = p$.
2. If $f''(p) > 0$, then $f(x)$ has a LOCAL MIN at $x = p$.
3. If $f''(p) = 0$, then this TEST FAILS. GO TO FIRST DERIVATIVE TEST.

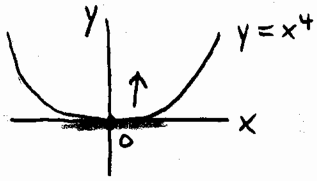
MEMORIZE

4/10/01 Tues, Lecture

E.g., Failure with the SECOND DERIVATIVE TEST.

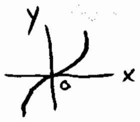


$f'(0) = 0$
$f''(0) = 0$
Neither a local max or min



$f'(0) = 0$
$f''(0) = 0$
A local min

<u>Example</u>. $f(x) = x^3$.



4/11/01 W Lecture

$f'(x) = \frac{d}{dx}(x^3) = 3x^2$

$f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$ CRITICAL POINT

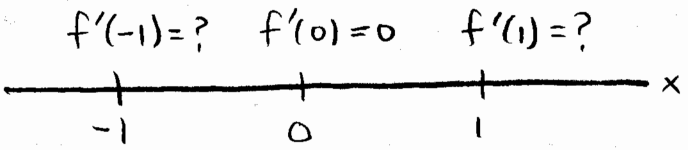
<u>2nd Deriv. Test</u>:

$f''(x) = \frac{d}{dx}(3x^2) = 3(2x) = 6x$

$f''(0) = 6(0) = 0$ TEST FAILS ∴ Try 1st Deriv. Test

plug in CRITICAL POINT

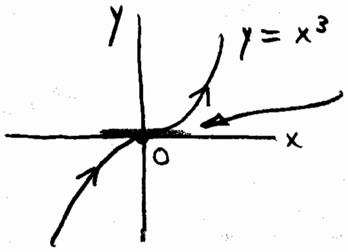
<u>1st Deriv. Test</u>:



Choose a sample point "right before" $x=0$, like $x=-1$

Choose a sample point "right after" $x=0$, like $x=1$

$$f'(-1) = 3(-1)^2 = 3 > 0 \Rightarrow f'(x) > 0 \text{ right before } x=0$$
$$f'(1) = 3(1)^2 = 3 > 0 \Rightarrow f'(x) > 0 \text{ right after } x=0$$



Called a "seat" at $x=0$

4/11/01
W
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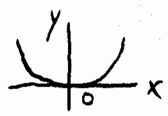
$$\boxed{f(x) \text{ has NEITHER a local max or min at } x=0.}$$

Example. $f(x) = x^2$.



$$f'(x) = \frac{d}{dx}(x^2) = 2x$$

$$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

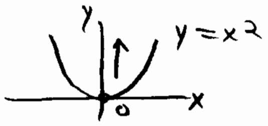
CRITICAL POINT

2nd Deriv. Test:

$$f''(x) = \frac{d}{dx}(2x) = 2$$

$$f''(0) = 2 > 0 \Longrightarrow \boxed{f(x) = x^2 \text{ has a LOCAL MIN at } x = 0}$$

plug in CRITICAL POINT



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Concavity and the Test for an Inflection Point

The following is a definition of inflection point, which also serves as a test for an inflection point.

> $x = p$ is an INFLECTION POINT of $f(x)$ if
>
> ① $f''(p) = 0$
>
> ② concavity changes at $x = p$ where either
>
> ⓐ $f''(x) > 0$ "right before" $x = p$ and $f''(x) < 0$ "right after" $x = p$
>
> or ⓑ $f''(x) < 0$ "right before" $x = p$ and $f''(x) > 0$ "right after" $x = p$

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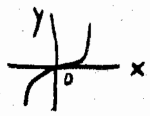
4/11/01 W Lecture

4/13/01 F lecture

Example. $f(x) = x^3$.

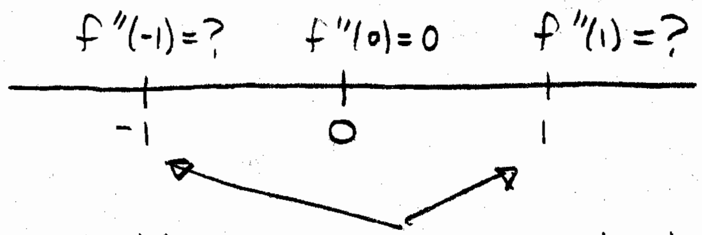


$$f'(x) = \frac{d}{dx}(x^3) = 3x^2$$

$$f''(x) = \frac{d}{dx}(3x^2) = 3(2x) = 6x$$

$$f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$$

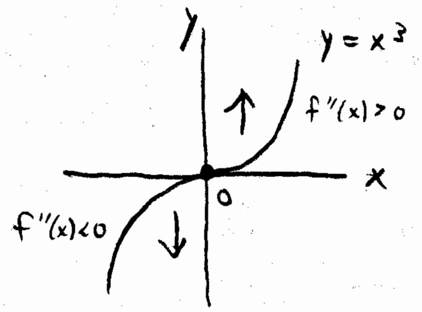
CANDIDATE FOR INFLECTION POINT



These are sample points chosen "right before" and "right after" $x=0$

$$\left.\begin{array}{l} f''(-1) = 6(-1) = -6 < 0 \Rightarrow f''(x) < 0 \text{ right before } x=0 \\ f''(1) = 6(1) = 6 > 0 \Rightarrow f''(x) > 0 \text{ right after } x=0 \end{array}\right\}$$

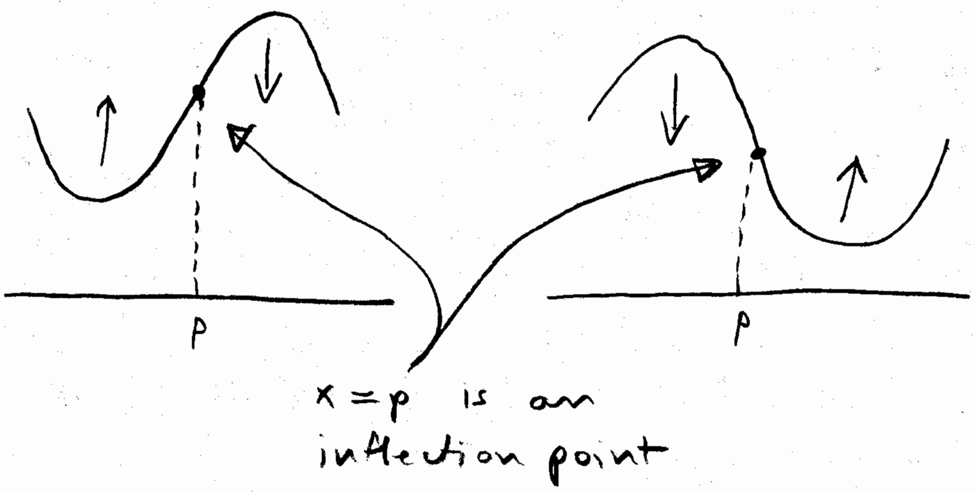
Change in concavity at $x=0$



$\therefore$ $\boxed{x=0 \text{ is an INFLECTION POINT of } f(x)}$

4/13/01
F
Lecture

<u>Example</u>. Inflection points are always between local max and min:



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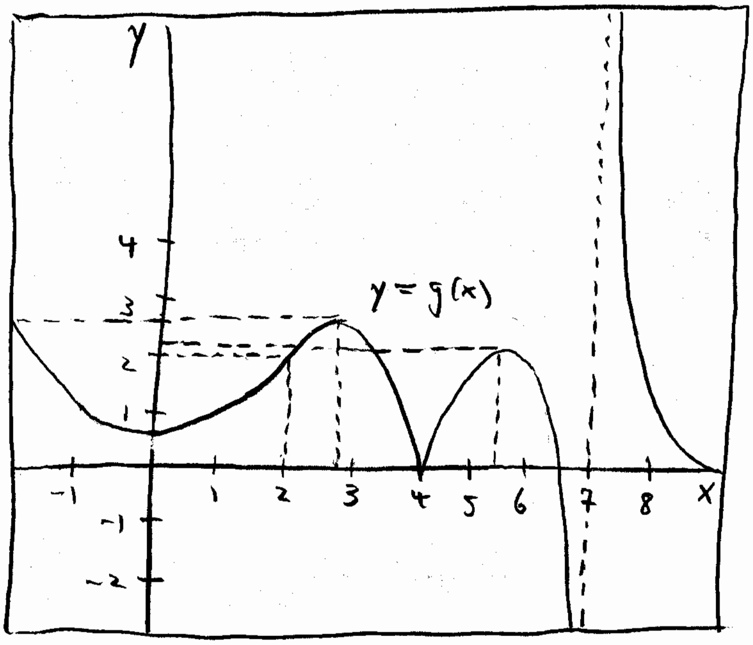
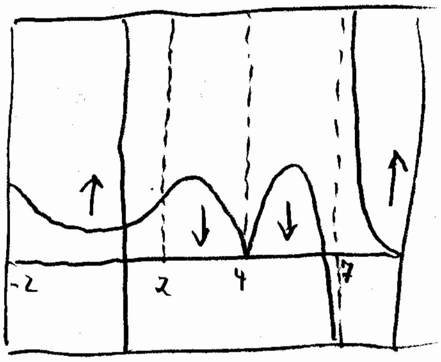
Examples.

1. (HW Exercise 2, p. 292, Sect. 4.3)

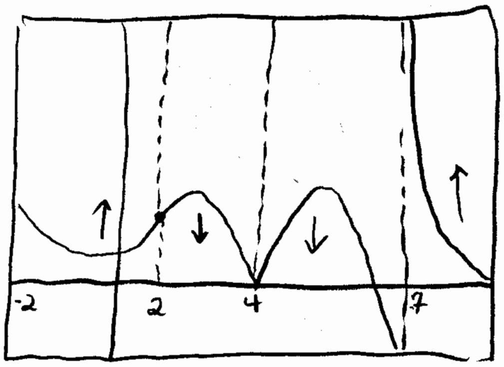


(a) Where is g concave up? (Give open intervals.)
(b) Where is g concave down? (Give open intervals.)
(c) What are the inflection points of g?

(a) $\boxed{(-2, 2) \cup (7, 9)}$ (b) $\boxed{(2,4) \cup (4,7)}$



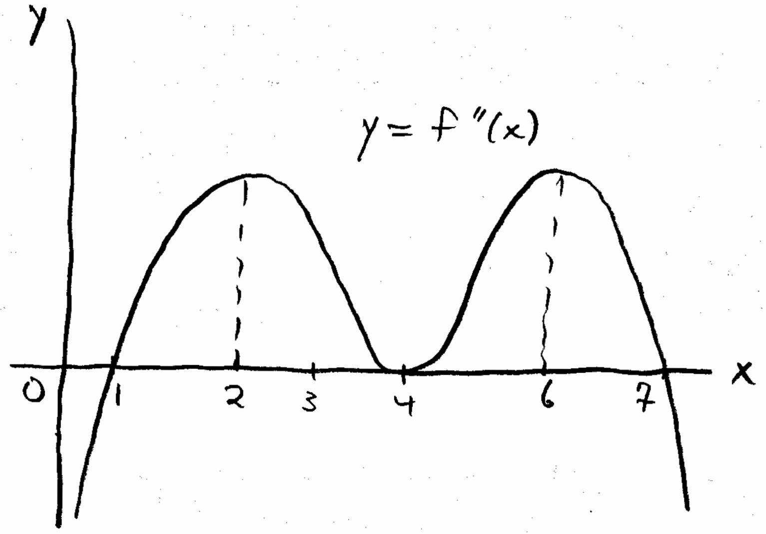
(c) $\boxed{x = 2}$



$x = 2$
(or $(x, y) = (2, 2)$)
is a point of inflection where g changes from concave up to concave down

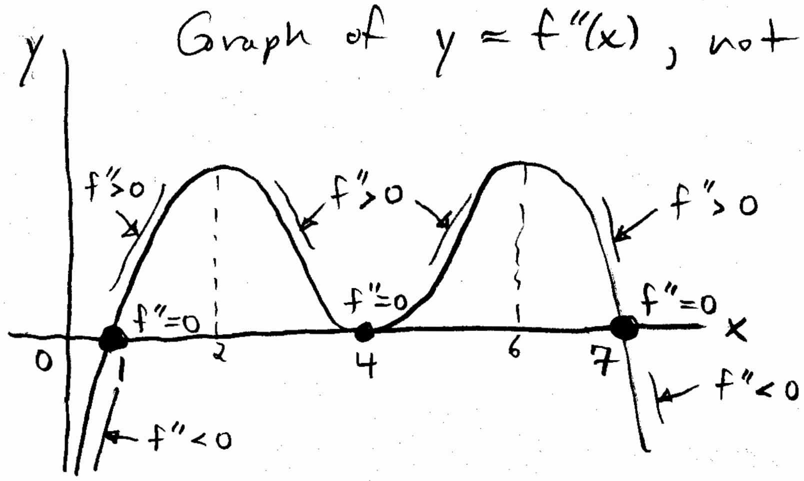
$x = 7$
is <u>not</u> a point of inflection, even though g changes concavity from concave down to concave up

2. (HW Exercise 5, p. 292, Sect. 4.3.)



What are the inflection points of f?

Graph of $y = f''(x)$, not $y = f(x)$!



Inflection pts are (at) $\boxed{x = 1 \text{ and } x = 7}$ where f'' changes sign (from neg. to pos. or pos. to neg.)

($x = 4$ is <u>not</u> an inflection point, since f'' does not change sign!)

3. Let

$$f(x) = x^4 + 12x^3 + 16x^2$$

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Find all local maxima and minima.

$f(x) = x^4 + 12x^3 + 16x^2$

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$f'(x) = 4x^3 + 36x^2 + 32x$

$f''(x) = 12x^2 + 72x + 32$

Set $f'(x) = 0$ and solve for x:

4/16/01 M Lecture

$f'(x) = 0 \implies 4x^3 + 36x^2 + 32x = 0$

$\implies 4x(x^2 + 9x + 8) = 0$

$\implies 4x(x+8)(x+1) = 0$

$\implies x = 0, \ x = -1, \ x = -8$

3 CRITICAL POINTS
(candidates for being local max's and min's)

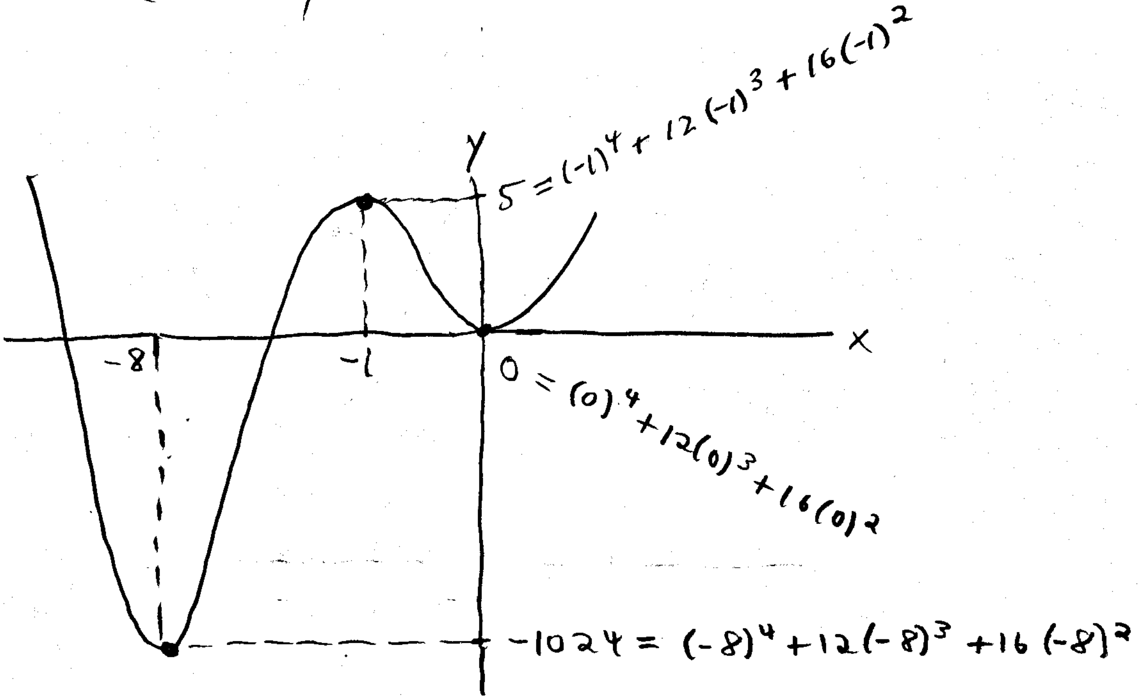
2nd Deriv. Test:

$f''(0) = 12(0)^2 + 72(0) + 32 = 32 > 0 \Rightarrow$ f(x) has a LOCAL MIN at $x = 0$

$f''(-1) = 12(-1)^2 + 72(-1) + 32 = -28 < 0 \Rightarrow$ f(x) has a LOCAL MAX at $x = -1$

$f''(-8) = 12(-8)^2 + 72(-8) + 32 = 224 > 0 \Rightarrow$ f(x) has a LOCAL MIN at $x = -8$

Sketch of $y = x^4 + 12x^3 + 16x^2$:



3. (HW Exercise 7, p. 292, Sect. 4.3.)

Let

$$f(x) = x^6 + 192x + 17$$

$f' > 0$ $\qquad$ $f' < 0$

(a) Where is f increasing and decreasing?

$f' = 0$ $\qquad$ $f' = 0$

(b) Where are the local max and min of f?

$f'' > 0$ $\qquad$ $f'' < 0$

(c) Where is f concave up and down?
What are the inflection points of f?

$f'' = 0$

$$f(x) = x^6 + 192x + 17$$

$$f'(x) = 6x^5 + 192$$

$$f''(x) = 30x^4$$

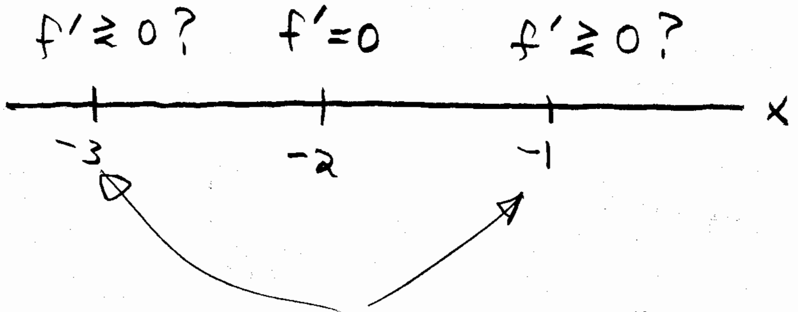
Set $f'(x) = 0 \Rightarrow 6x^5 + 192 = 0$

$\Rightarrow 6x^5 = -192$

$\Rightarrow x^5 = -\frac{192}{6} = -32$

$\Rightarrow x = -2$ CRITICAL POINT (candidate for being a local max or a local min)

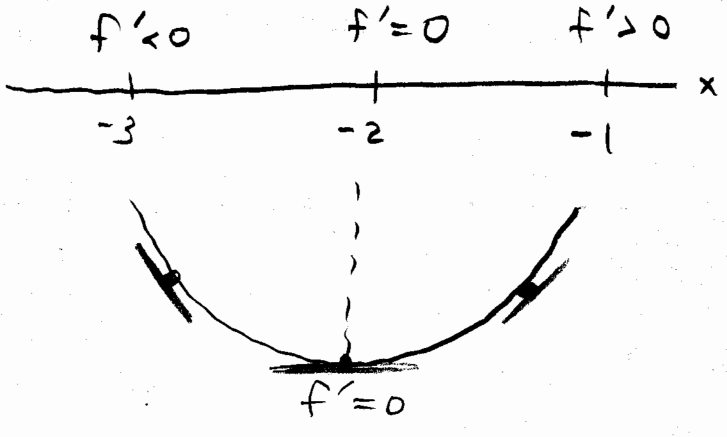
Check to see if f has a local max or min using the FIRST DERIVATIVE TEST :



Sample points "right before" and "right after" $x = -2$

$$f'(-3) = 6(-3)^5 + 192 = 6(-243) + 192$$
$$= -1458 + 192 = \underline{-1266 < 0}$$

$$f'(-1) = 6(-1)^5 + 192 = 6(-1) + 192$$
$$= -6 + 192 = \underline{186 > 0}$$



LOCAL MIN

4/16/01
M
Lecture

$\therefore$ $\boxed{\text{f has one local min at } x=-2 \text{ and nothing else, and so no local max.}}$

The LOCAL MIN VALUE of f is then

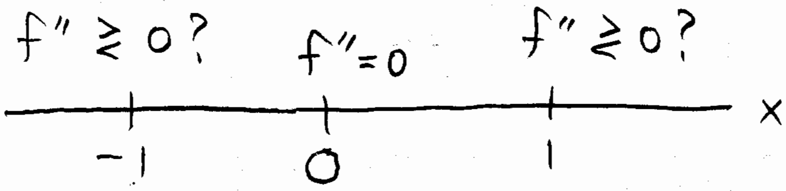
$$f(-2) = (-2)^6 + 192(-2) + 17$$
$$= 64 - 384 + 17$$
$$= -303$$

plug in pt at which local min occurs

Check to see if f has any inflection points:

4/16/01 M Lecture

Set $f''(x) = 0 \Rightarrow 30x^4 = 0$
$\Rightarrow x^4 = 0$
$\Rightarrow x = 0$ Candidate for being an INFLECTION POINT

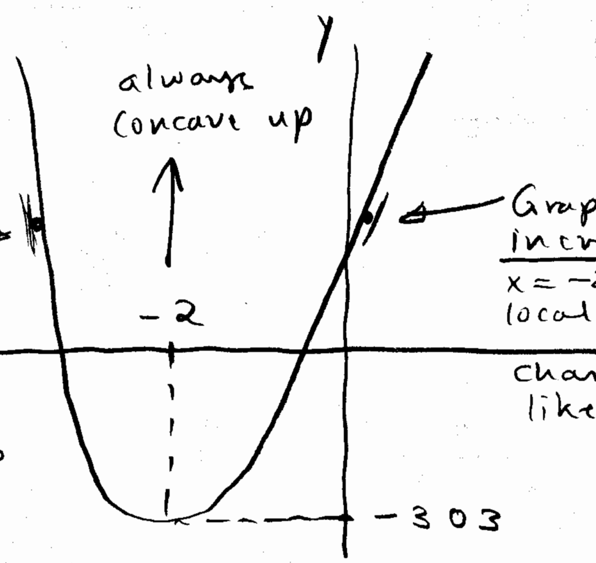


$$f''(-1) = 30\,(-1)^4 = 30\,(1) = \underline{\underline{30 > 0}}$$

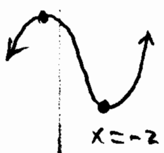
$$f''(1) = 30\,(1)^4 = 30\,(1) = \underline{\underline{30 > 0}}$$

$\therefore$ $x=0$ is NOT an inflection point of f, since f'' does not change sign around $x=0$!

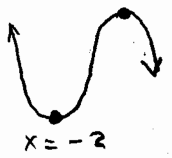
SKETCH OF $f(x)$:



Graph is <u>always decreasing</u> before $x=-2$, since no local max to change that, like so



Graph is <u>always increasing</u> after $x=-2$, since no local max to change that, like so



4/16/01
M
Lecture

SUMMARY :

$\boxed{\begin{array}{l} f \text{ has one local min at } x=-2, \\ f \text{ has no local max,} \\ f \text{ has no inflection points,} \\ f \text{ is decreasing on } (-\infty,-2) \\ f \text{ is increasing on } (-2,\infty) \\ f \text{ is concave up on } (-\infty,\infty) \end{array}}$

4/17/01
Tues.
Lecture

Finding Global (or Absolute) Maxima and Minima

[We now will focus primarily on Section 4.2. Previously we were focusing on Section 4.3.]

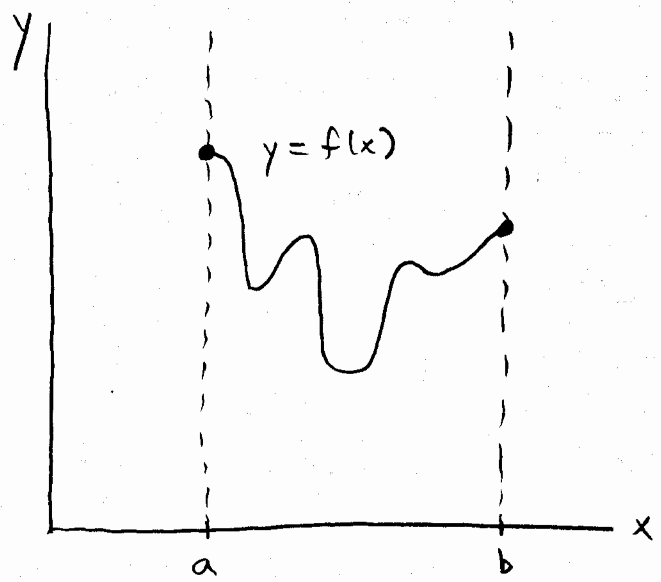
We will consider

CONTINUOUS FUNCTIONS

(i.e., no holes or breaks or jumps in their graphs) on

CLOSED INTERVALS

(i.e., $a \le x \le b$ or $[a, b]$ versus $a < x < b$ or (a, b)).



We will look for the following:

1. THE global (or absolute) maximum value of f, which may be achieved at one or more locations in the graph of $y = f(x)$ on $[a, b]$.

2. THE global (or absolute) minimum value of f, which may be achieved at one or more locations in the graph of $y = f(x)$ on $[a, b]$.

We are given a guarantee by the following theorem:

THEOREM. Let f be a continuous function on the closed interval $[a, b]$. Then f will have exactly one global maximum value and exactly one global minimum value.

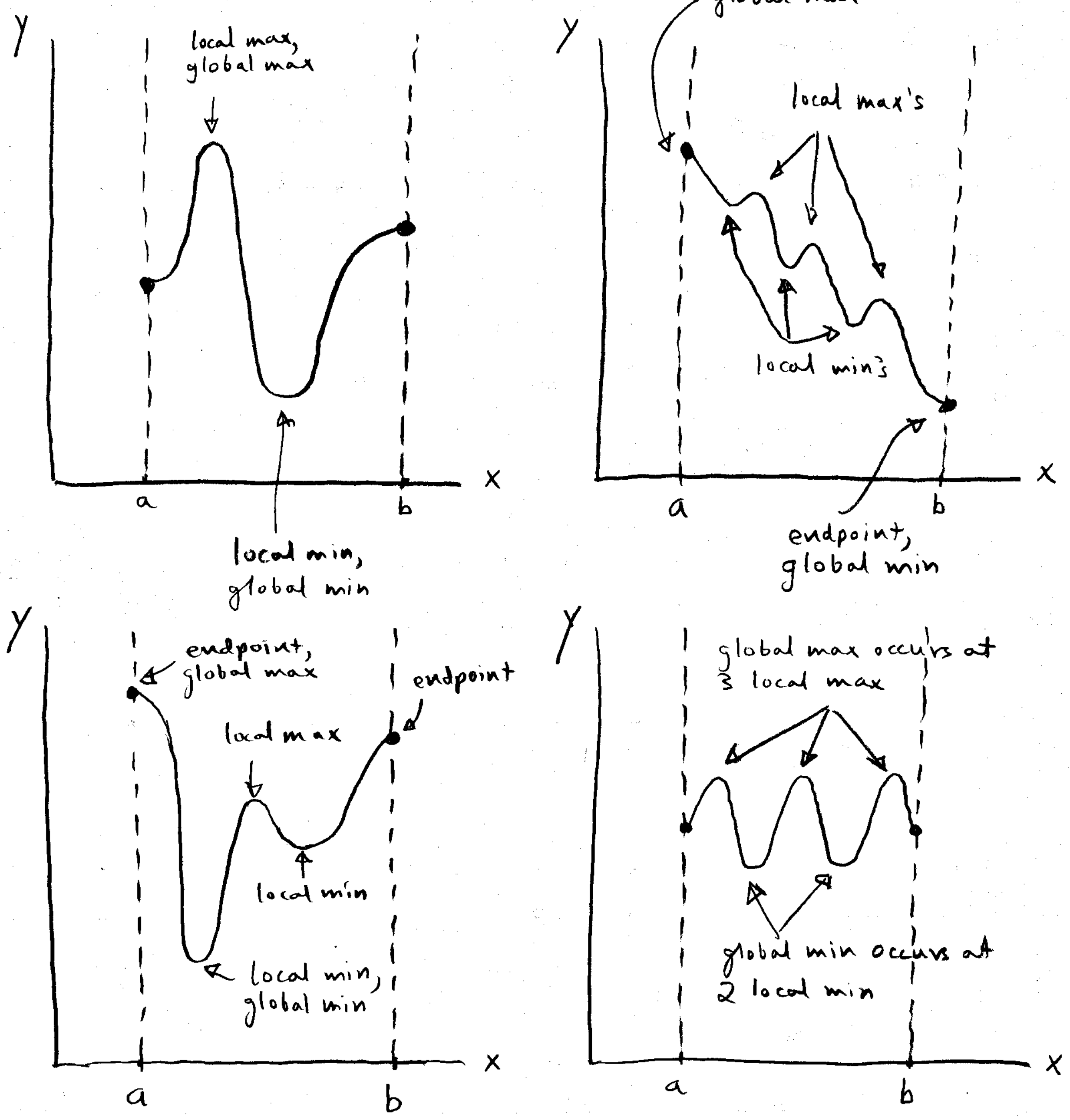
Remark. f may have many or no local maximum values or local minimum values.

RULE OF THUMB.

1. Global maximum value of f
   - will occur at one (or more) of the local maxima of f or at one (or both) of the endpoints, $x = a$ and $x = b$.

2. Global minimum value of f
   - will occur at one (or more) of the local minima of f <u>or</u> at one (or both) of the endpoints, $x=a$ and $x=b$.

E.g.,



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We are given a <u>recipe</u> to find locations of global max and min of f when f is continuous on a closed interval $[a,b]$

<u>RECIPE</u>.

<u>STEP 1</u>. Find all critical points of $f(x)$:

Where $f'(x) = 0$ (or $f'(x) = \infty$).
Optional: Determine if critical pts correspond to local max and/or min.

<u>STEP 2</u>. Evaluate $f(x)$ at EVERY critical point AND at BOTH endpoints $(x=a,\ x=b)$.

<u>STEP 3</u>. Global max value = largest value of $f(x)$ from STEP 2

Global min value = smallest value of $f(x)$ from STEP 2

Example. (HW Exercise 38, p. 280.)

Find absolute (= global) max/min values of

$$f(x) = x^3 - 12x + 1 \quad \text{on} \quad [-3, 5]$$

Note: $f(x)$ = polynomial $\Rightarrow$ $f(x)$ is continuous on $(-\infty, \infty)$ $\Longrightarrow$ $f(x)$ is certainly continuous on $[-3, 5]$.

STEP 1. Critical Points:

$f'(x) = 3x^2 - 12$

$$\begin{aligned} f'(x) = 0 &\Longrightarrow 3x^2 - 12 = 0 \\ &\Longrightarrow 3x^2 = 12 \\ &\Longrightarrow x^2 = 4 \\ &\Longrightarrow x = -2,\ x = 2 \end{aligned}$$

STEP 2. Evaluate $f(x)$ at $x = \underbrace{-2, 2}_{\text{CRITICAL POINTS}}, \underbrace{-3, 5}_{\text{ENDPOINTS}}$

$f(x) = x^3 - 12x + 1$:

$f(-2) = (-2)^3 - 12(-2) + 1 = 17$

$f(2) = (2)^3 - 12(2) + 1 = -15 \longleftarrow$ SMALLEST

$f(-3) = (-3)^3 - 12(-3) + 1 = 10$

$f(5) = (5)^3 - 12(5) + 1 = 66 \longleftarrow$ LARGEST

STEP 3. Absolute (= global) max/min values of $f(x)$:

4/17/01 Tues, Lecture

$$\boxed{\begin{array}{ll} f(2) = -15 & \text{absolute min value at } x = 2 \\ f(5) = 66 & \text{absolute max value at } x = 5 \end{array}}$$