Lecture

Section 2.6. Tangents, Velocities, and Other Rates of Change.

This section refers to Section 2.1, which we SKIPPED. The material in Section 2.1 is essentially covered, in a formal way, in this section.

QUESTION: How does one measure speed at an instant in time?

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The answer to this is related to the "derivative," introduced in the next section.

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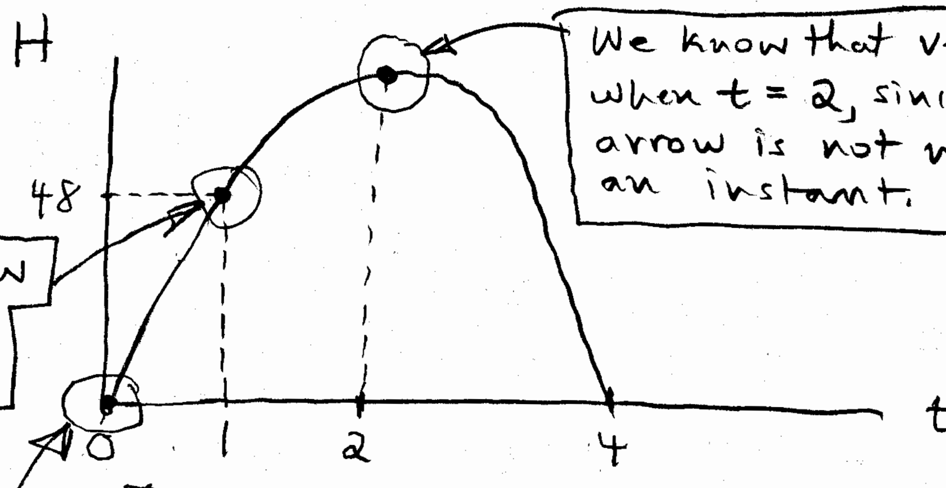
Example. (SEE HW Exercise 16, p. 150.)

If an arrow is shot upward on the moon with a velocity (= speed + direction) of 64 m/s at time $t=0$, its height (in meters) after t seconds is given by $H = 64t - 16t^2$.

(a) Find the velocity of the arrow after 1 s.



We know that velocity = 0 when $t = 2$, since the arrow is not moving for an instant.

BUT HOW do we compute velocity at $t = 1$?

The graph of $H = 64t - 16t^2 = -16t^2 + 64t$ is an "upside down" parabola with the t-intercepts $t = 0$ and $t = 4$;

$$H = 0 \Rightarrow 64t - 16t^2 = 0$$
$$\Rightarrow 16t(4 - t) = 0$$
$$\Rightarrow t = 0, \quad t = 4$$

We know velocity = 64 when $t = 0$, since that is given.

We also know how to compute the "average velocity" of the arrow from time $t = 0$ to $t = 1$:

$$\text{average velocity from } t = 0 \text{ to } t = 1 = \frac{\text{change in distance}}{\text{change in time}}$$
$$= \frac{48 - 0}{1 - 0} = 48$$

HOW do we compute the velocity at $t=1$?

Let us try to compute this velocity <u>at an instant in time</u>, just as we would compute average velocity <u>over an interval of time</u>:

$$\text{instantaneous velocity at } t=1 = \frac{48-48}{1-1} = \frac{0}{0}$$

NONSENSICAL RESULT!
But maybe if we take some sort of limit we will get an answer.

<u>Terminology, Notation, and Definitions</u>

(Instantaneous) Velocity
= speed (magnitude) + direction (up/down, right/left)
– refers to speed (with a direction) <u>at an instant in time</u>

Average Velocity
= speed (with a direction) <u>over an interval of time</u>

Let $s(t)$ = distance traveled as a function of time

Then

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$$\boxed{\left.\begin{array}{l}\text{average velocity}\\ \text{over the interval}\\ a \le t \le b\end{array}\right\} = \frac{\text{change in distance}}{\text{change in time}} = \frac{s(b)-s(a)}{b-a}}$$

Now, again, defining instantaneous velocity in a similar way is fruitless:

$$\left.\begin{array}{l}\text{instantaneous}\\ \text{velocity at}\\ t = a\end{array}\right\} = \left\{\begin{array}{l}\text{instantaneous}\\ \text{velocity over}\\ \text{the interval } a \le t \le a\end{array}\right.$$

$$= \frac{s(a)-s(a)}{a-a}$$

$$= \frac{0}{0}$$

So, we do what we did in Section 2.2 when we plugged $x=a$ into $f(x)$ and obtained $f(a) = \frac{0}{0}$; We see what value $\dfrac{s(t)-s(a)}{t-a}$ approaches as t approaches a. So, we define instantaneous velocity this way:

$$\boxed{\left.\begin{array}{l}\text{instantaneous}\\ \text{velocity at}\\ t=a\end{array}\right\} = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}}$$

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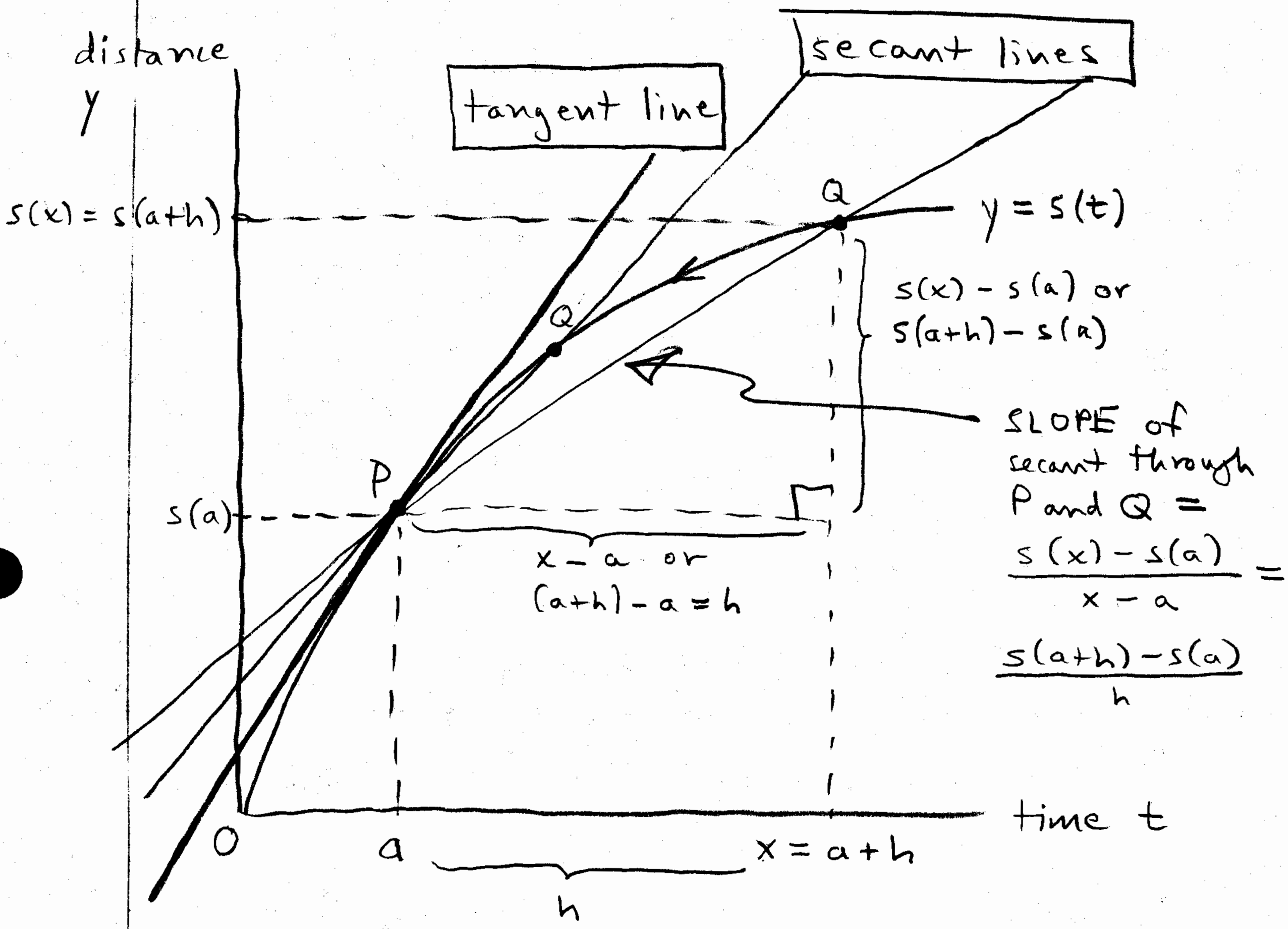
Graphical Interpretations of Average and Instantaneous Velocity



Let x get closer and closer to $a \Rightarrow$ the distance h approaches 0

$$\left.\begin{array}{l}\text{average velocity}\\ \text{over the interval}\\ a \le t \le x\end{array}\right\} = \frac{s(x)-s(a)}{x-a}$$

$$= \frac{s(a+h)-s(a)}{h}$$

= SLOPE of secant through P and Q

$$\left.\begin{array}{l}\text{instantaneous}\\ \text{velocity at}\\ t=a\end{array}\right\} = \lim_{x\to a} \frac{s(x)-s(a)}{x-a}$$

$$= \lim_{h\to 0} \frac{s(a+h)-s(a)}{h}$$

$$= \left\{\begin{array}{l}\text{LIMIT OF THE}\\ \text{SLOPES of the}\\ \text{secants through P}\\ \text{and Q as Q}\\ \text{approaches P}\end{array}\right.$$

$$= \left\{\begin{array}{l}\text{SLOPE of tangent}\\ \text{through P (or at}\\ x=a)\end{array}\right.$$

∴ Ave. Vel. = slope of secant

Inst. Vel. = slope of tangent

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Example. (HW Exercise 16, p. 150.)

If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by

$$H = 58t - 0.83t^2.$$

(a) Find the velocity of the arrow after 1 s.
(b) Find the velocity of the arrow when $t = a$.
(c) When will the arrow hit the moon?
(d) With what velocity will the arrow hit the moon?

(a) Let $v(t)$ = (instantaneous) velocity at time t.

$$= 58t - 0.83t^2$$

$$v(1) = \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1}$$

$$= \lim_{t \to 1} \frac{(58t - 0.83t^2) - [58(1) - 0.83(1)^2]}{t - 1}$$

$$= \lim_{t \to 1} \frac{58t - 0.83t^2 - 58 + 0.83}{t - 1}$$

$$= \lim_{t \to 1} \frac{58(t-1) - 0.83(t^2 - 1)}{t - 1}$$

$$= \lim_{t\to 1} \frac{58\cancel{(t-1)} - 0.83\cancel{(t-1)}(t+1)}{\cancel{t-1}}$$

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$$= \lim_{t\to 1} \left[58 - 0.83\,(t+1)\right]$$

$$= 58 - 0.83\,(1+1)$$

$$= \boxed{56.34 \text{ m/s}}$$

(b) $$V(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}$$

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instead of $s(t)$, use $s(a+h)$

instead of $t-a$, use $(a+h)-a = h$

$$s(t) = 58t - 0.83t^2 \implies$$
$$s(a+h) = 58(a+h) - 0.83\,(a+h)^2$$

$$= \lim_{h\to 0} \frac{[58(a+h) - 0.83(a+h)^2] - [58a - 0.83a^2]}{h}$$

$$58(a+h) - 0.83\,(a+h)^2 = 58a + 58h - 0.83\,(a^2 + 2ah + h^2)$$
$$= 58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2$$

$$= \lim_{h\to 0} \frac{\cancel{58a} + 58h - \cancel{0.83a^2} - 1.66ah - 0.83h^2 - \cancel{58a} + \cancel{0.83a^2}}{h}$$

$$= \lim_{h \to 0} \frac{58h - 1.66ah - 0.83h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(58 - 1.66a - 0.83h)}{\cancel{h}}$$

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$$= \lim_{h \to 0} (58 - 1.66a - 0.83h)$$

$$= 58 - 1.66a - 0.83(0)$$

$$= \boxed{58 - 1.66a \quad m/s}$$

(c) When the arrow hits the moon, distance H of the arrow from the moon is 0. So, set $H = 0$ and solve for t:

$$H = 0 \Rightarrow 58t - 0.83t^2 = 0$$
$$\Rightarrow t(58 - 0.83t) = 0$$
$$\Rightarrow t = 0 \quad \text{or} \quad \boxed{t = \frac{58}{0.83}} s$$

When the arrow LEAVES the moon

This must be when the arrow HITS the moon

(d) The velocity with which the arrow hits the moon, which is at time $t = 58/0.83$ s, according to Part (c), can be given by the formula obtained in Part (b):

$$v(a) = 58 - 1.66\,a \implies$$

$$v\left(\frac{58}{0.83}\right) = 58 - 1.\overset{2}{\cancel{66}}\left(\frac{58}{\cancel{0.83}}\right)$$

$$= 58 - 2(58)$$

$$= \boxed{-58 \text{ m/s}}$$

negative since direction of arrow is DOWN

Generalizing These Ideas for the Distance Function

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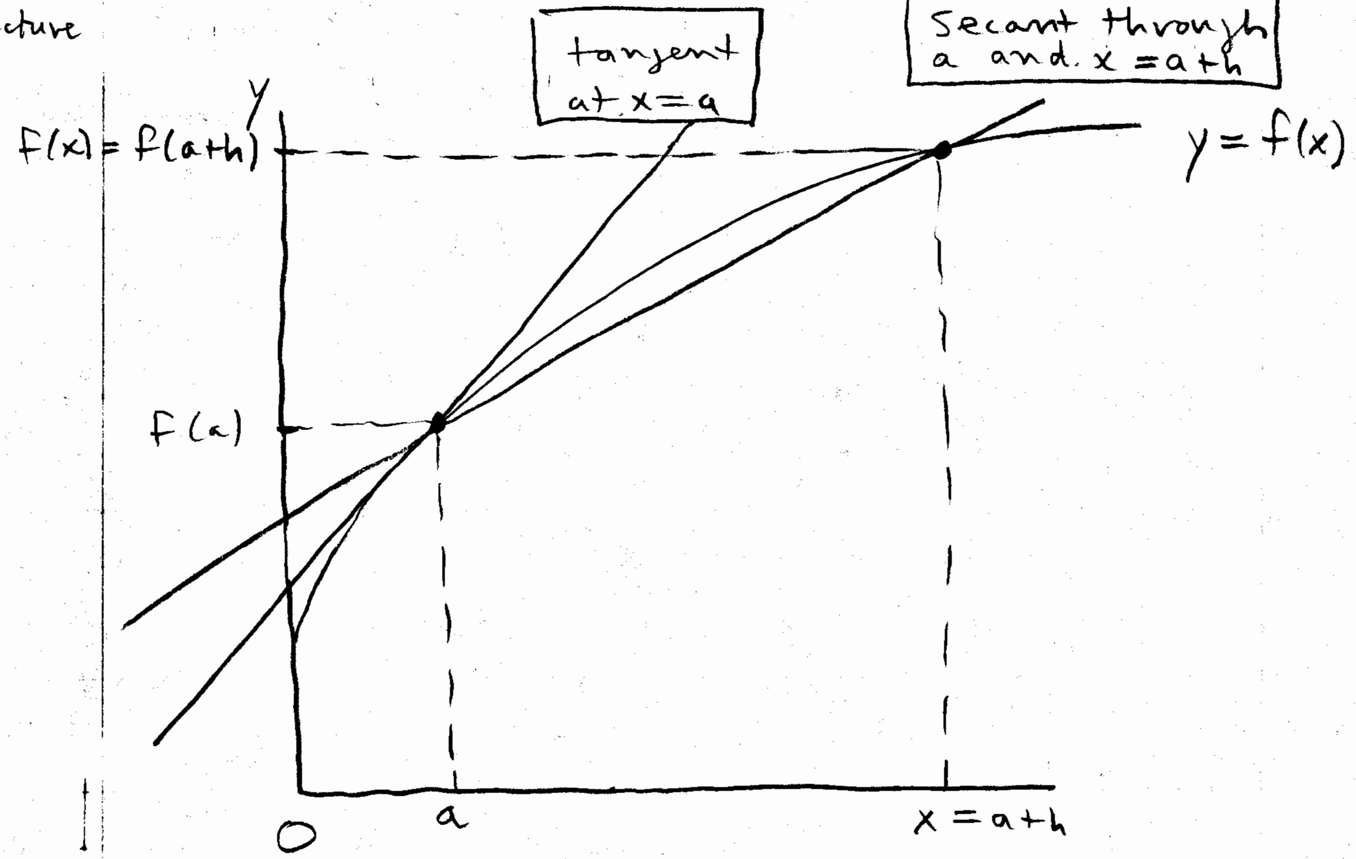
Let $f(x)$ be any function with graph



Terminology:

$$\frac{f(x)-f(a)}{x-a} = \frac{f(a+h)-f(a)}{h} \text{ : Called}$$

(1) average rate of change of f over the interval $a \le t \le x$ (or $a \le t \le a+h$)

or (2) difference quotient of f

or (3) slope of the secant through the points a and $x = a+h$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$: Called

(1) instantaneous rate of change of f at $x = a$

or (2) slope of tangent to curve $y = f(x)$ at $x = a$

or (2) slope of the curve $y = f(x)$ at $x = a$

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Examples.

1. (HW Exercise 7, p. 149.)

Find the equation of the tangent line to the curve

$y = \sqrt{x}$ at the point $(1,1)$.

Set $f(x) = \sqrt{x}$ and $a = 1$. Then

$$\left.\begin{array}{l}\text{Slope of tangent} \\ \text{line at } x=1\end{array}\right\} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$\quad\;\; \parallel$

$\;\; m_{tan}$

$$\begin{aligned}
m_{tan} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} \\
&= \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}
\end{aligned}$$

TRICK: Multiply top and bottom by $\sqrt{x} + 1$

$$\begin{aligned}
&= \lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \\
&= \lim_{x \to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} \\
&= \lim_{x \to 1} \frac{\sqrt{x}\sqrt{x} + \cancel{\sqrt{x}} - \cancel{\sqrt{x}} - 1}{(x - 1)(\sqrt{x} + 1)} \\
&= \lim_{x \to 1} \frac{\sqrt{x^2} - 1}{(x - 1)(\sqrt{x} + 1)} \\
&= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(\sqrt{x} + 1)} \\
&= \lim_{x \to 1} \frac{1}{\sqrt{x} + 1} \\
&= \frac{1}{\sqrt{1} + 1} = \frac{1}{1 + 1} = \frac{1}{2}
\end{aligned}$$

$\therefore$ $m_{tan} = \frac{1}{2}$ $(x_1, y_1) = (1, 1)$

$$y - y_1 = m_{tan}(x - x_1) \Longrightarrow$$

$$\boxed{y - 1 = \frac{1}{2}(x - 1)} \Longrightarrow$$

$$y - 1 = \frac{1}{2}x - \frac{1}{2} \Longrightarrow y = \frac{1}{2}x - \frac{1}{2} + 1 \Longrightarrow$$

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

2. (HW Exercise 8, p. 149.)

Find the equation of the tangent line to the curve

$$y = \frac{x}{1-x} \quad \text{at the point } (0, 0).$$

Set $f(x) = \dfrac{x}{1-x}$ and $a = 0$. Then

$$\left.\begin{array}{l}\text{Slope of tangent} \\ \text{line at } a = 0\end{array}\right\} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$\parallel$

m_{tan}

$$m_{tan} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$f(x) = \frac{x}{1-x}$$

$$f(0+h) = f(h) = \frac{h}{1-h}$$

$$f(0) = \frac{0}{1-0} = 0$$

$$= \lim_{h \to 0} \frac{\frac{h}{1-h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{1-h}}{h}$$

$$\frac{h}{1-h} \div h = \frac{h}{1-h} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{1-h} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{1}{1-h}$$

$$= \frac{1}{1-0} = \frac{1}{1} = 1$$

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$m_{tan} = 1 \qquad (x_1, y_1) = (0, 0)$

$$y - y_1 = m_{tan}(x - x_1) \implies$$

$$y - 0 = 1 \cdot (x - 0) \implies \boxed{y = x}$$